Some applications of interpolation determinants

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1 Preliminaries

Proposition 1 (Hadamard's Inequality)**.**

$$|\det(A)| \le \prod_i ||v_i|| \tag{1}$$

Proposition 2 (Schwarz' Lemma)**.** *Let f be holomorphic in the disk of radius R, with a zero of multiplicity n at 0, and $r \le R$. Then $|f(z)|_r \le \left(\frac{r}{R}\right)^n |f(z)|_R$.*

Proof. Define

$$g(z) = \begin{cases} \frac{f(z)}{z^n}, & z \ne 0 \\ f^{(n+1)}(0), & z = 0 \end{cases} \tag{2}$$

This is holomorphic on the disk of radius R, so by the maximum modulus principle,

$$|g(z)|_R \le \frac{|f(z)|_R}{R^n} \tag{3}$$

so

$$|f(z)|_r \le r^n |g(z)|_r \tag{4}$$

$$\le r^n |g(z)|_R \tag{5}$$

$$\le \left(\frac{r}{R}\right)^n |f(z)|_R \tag{6}$$

□

Proposition 3. *Let $0 < r \le R$, $f_1, \dots, f_N$ be analytic on the disk of radius R, and $\zeta_1, \dots \zeta_N$ lie in the disk of radius r. Then*

$$\Delta = |\det\left(f_i(\zeta_j)\right)| \le \left(\frac{r}{R}\right)^{\frac{N^2-N}{2}} N^{\frac{N}{2}} \prod_i |f_i(z)|_R \tag{7}$$

Proof. Consider $F(z) = \det\left(f_i(\zeta_j z)\right)$, so $\Delta = F(1)$.

F has a zero of multiplicity $\frac{N^2-N}{2}$ at 0, as if we expand the f_i at zero into their taylor series $\sum a_j^{(i)} z^j$, and consider a 2×2 minor of $F(z)$ in which the terms are of the same order k, we have

$$\det \begin{pmatrix} a_k^{(i_1)}(\zeta_{j_1}z)^k & a_k^{(i_1)}(\zeta_{j_2}z)^k \\ a_k^{(i_2)}(\zeta_{j_1}z)^k & a_k^{(i_2)}(\zeta_{j_2}z)^k \end{pmatrix} = 0 \tag{8}$$

So any surviving product in this infinite expansion can not have any two orders of factors repeated, so as $\sum_{m=0}^{N-1} m = \frac{m(m-1)}{2}$, we have

$$F(z) = \sum_{i=\frac{N^2-N}{2}}^{\infty} c_i z^i \tag{9}$$

The result then follows by Schwarz' lemma and Hadamard's inequality.

$$\Delta \leq |F|_1 \leq \left(\frac{r}{R}\right)^{\frac{N^2-N}{2}} |F|_{\frac{R}{r}} \tag{10}$$

Each entry $f_i\left(\frac{R}{r}\zeta_j\right)$ of $|F|_{\frac{R}{r}}$ is bounded by $|f_i|_R$, so the length of each row as a vector is $\leq \sqrt{n}|f_i|_R$.

□

Other such zero multiplicities will be similar.

2 Interpolation Determinants

2.1 Mahler measure

We consider the $N \times N$ determinant

$$\Delta = \det\left(\alpha^{ij}\right) \tag{11}$$

2.1.1 Zero estimate

If $\Delta = 0$, then there exist $a_1, \ldots, a_N$ such that

$$a_1 + a_2 + \ldots + a_N = 0 \tag{12}$$

$$a_1 + a_2\alpha^1 + \cdots + a_N\alpha^{N-1} = 0 \tag{13}$$

$$\vdots \tag{14}$$

$$a_1 + a_2\alpha^{N-1} + \cdots + a_N^{(N-1)^2} = 0 \tag{15}$$

So we have a polynomial $P(X)$ with roots $1, \alpha, \alpha^2 \ldots, \alpha^{N-1}$ of degree $N-1$.

2.1.2 Arithmetical lower bound

Consider the expansion of this determinant, a polynomial of degree $\leq N^3$, with length $\leq N!$.

$$P(\alpha) = \Delta \tag{16}$$

So since $P(\alpha) \neq 0$, Liouville's inequality gives, with M the mahler measure of α,

$$|P(\alpha)| \geq (N!)^{1-d} M^{-N^3} \tag{17}$$

2.1.3 Analytic upper bound

We apply the lemma to the functions $e^{zj \log \alpha}$, $j = 0, \dots, N-1$, at points $0, \dots, N-1$, taking our radius to be $2N$.

$$\begin{aligned} |\Delta| &\leq 2^{-\frac{N^2-N}{2}} N! \prod_j \left|\alpha^{jz}\right|_{2N} && (18) \\ &\leq 2^{-\frac{N^2-N}{2}} N! \alpha^{N^3} && (19) \\ &\leq 2^{-\frac{N^2-N}{2}} N! M^{N^3} && (20) \\ & && (21) \end{aligned}$$

2.1.4 Putting it together

So

$$(N!)^{1-d} M^{-N^3} \leq 2^{-\frac{N^2-N}{2}} N! M^{N^3} \tag{22}$$

Taking logs, we have

$$N^2 \frac{\log 2}{2} \leq dN \log N + 2N^3 \log M + N \frac{\log 2}{2} \tag{23}$$

Set $N = [4d \log d]$, then we have

$$\begin{aligned} (8 \log 2) d^2 (\log d)^2 &\leq 4d^2 (\log d)^2 + 128 d^3 (\log d)^3 \log M && (24) \\ &\quad + O(d^2 \log d \log \log d) && (25) \\ \frac{8 \log 2 - 4}{128 d \log d} + O\left(\frac{\log \log d}{d (\log d)^2}\right) &\leq \log M && (26) \\ \frac{1}{83 d \log d} &\leq \log M \text{ (for sufficiently large d)} && (27) \\ 1 + \frac{1}{83 d \log d} &\leq M && (28) \end{aligned}$$

2.2 Six exponentials

Proposition 4. *Suppose x_1, x_2, x_3, y_1, y_2, y_3 are real and linearly independent over $\mathbb{Q}$. Then at least one of $e^{x_i y_j}$ is non-integer/irrational/transcendental*

We consider the determinant

$$\Delta = \det\left(e^{(n_1 x_1 + n_2 x_2 + n_3 x_3)(m_1 y_1 + m_2 y_2 + m_3 y_3)}\right) \tag{29}$$

Where $n_1 + n_2 + n_3 \leq N$, $m_1 + m_2 + m_3 \leq N$, an $L \times L$ matrix with $cN^3 \leq L \leq \tilde{c} N^3$

2.2.1 Zero estimate

The functions $e^{(n_1x_1+n_2x_2+n_3x_3)x}$ are distinct by linear independence, and if Δ were zero there would exist L (by linear independence) roots of the following exponential sum (which cannot be identically zero as the periods are distinct)

$$\sum_{i=1}^{L} a_i e^{\alpha_i x} = 0 \tag{30}$$

$$\sum_{i=1}^{L-1} \frac{a_i}{a_L} e^{(\alpha_i-\alpha_L)x} + 1 = 0 \tag{31}$$

By Rolle's theorem, if this has L roots, then

$$\sum_{i=1}^{L-1} \frac{a_i(\alpha_i-\alpha_L)}{a_L} e^{(\alpha_i-\alpha_L)x} \tag{32}$$

has at least $L-1$ roots, which by the IH does not hold.
Base case: $ae^{\alpha x} \neq 0$ for any x.
So Δ is non-zero.

2.2.2 Bounds

Then apply the lemma with $f_i = e^{(n_1^{(i)}x_1+n_2^{(i)}x_2+n_3^{(i)}x_3)z}$, so

$$\Delta \leq 2^{-\frac{c^2N^6-cN^3}{2}} N! \prod |e^{(n_1^{(i)}x_1+n_2^{(i)}x_2+n_3^{(i)}x_3)z}|_{2\tilde{c}N} \tag{33}$$

$$\leq 2^{-\frac{c^2N^6-cN^3}{2}} N! e^{2\max\{|x_iy_j|\}\tilde{c}N^5} \tag{34}$$

$$\tag{35}$$

Now if all $e^{x_iy_j}$ were integers, we would have a lower bound of 1 for $|\Delta|$.

If they were rationals, with total denominator D, then multiplying each row by D^{N^2} would clear the denominators, so we have a lower bound of $D^{-\tilde{c}N^5}$ for Δ.

And if they were algebraic, with denominator D, we would have

$$1 \leq \left|\prod_{i=1}^{f}\left(D^{\tilde{c}N^5}\Delta\right)^{\sigma_i}\right| \leq \left|D^{\tilde{c}N^5}\Delta\right| D^{(f-1)\tilde{c}N^5} N!^{f-1} \max_{i,j}\{\text{House}\,(e^{x_iy_j})\}^{(f-1)\tilde{c}N^5} \tag{36}$$