Diophantine storage and a conjectural bivariate polynomial injection from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{Q}$

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February 12, 2019

Diophantine storage

Definition ([2] Diophantine storage (resp. positive existnetial))**.** *Let (M, L, ϕ) be a triple, with ϕ the interpretation of the language L in M.*

A formula is Diophantine if it is of the form $\exists x A$, where A is atomic.

(M', L', ϕ') is Diophantine in (M, L, ϕ) if there exists an injection from $M' \to M$ such that the image is Diophantine, ϕ-definable, and the inclusions $\phi'(P') \subseteq M^i$ are also Diophantine and ϕ-definable.

L admits a natural interpretation of ϕ^2 in the cartesian product M^2 of $\phi^2(P^{(i)}) = \phi(P^{(i)}) \times \phi(P^{(i)}) \subseteq M^{2i}$.

We say (M, L) admits diophantine storage if (M^2, L, ϕ^2) is diophantine in (M, L, ϕ).

Note, if $f : M^2 \to M$ is our diophantine injection, it may not be that $f(\phi^2(P)) = \phi(P)$.

> [Lew,Rosenberg, 1978] Motivation for these concepts includes extendible storage schemes for multidimensional arrays, pairing functions from recursive function theory, and, historically earliest, diagonal enumeration of Cartesian products

Definition ([2] To admit $\wedge, \vee$)**.** *To admit $\vee$ and $\wedge$ is to, for every p, q atomic, have some r atomic such that $R \vDash p \wedge$ (resp. $\vee$) $q \iff R \vDash r$, for example*

- $\mathbb{R} \vDash (f = 0 \vee g = 0) \iff \mathbb{R} \vDash fg = 0$
- $\mathbb{R} \vDash (f = 0 \wedge g = 0) \iff \mathbb{R} \vDash f^2 + g^2 = 0$

Proposition. *[2] Suppose R is a ring and L the language of rings, possibly with constants, interpreted in the natural way, which admits $\wedge$ and $\vee$, having:*

- *A disjoint finite cover $\{U_i\}_{i=1}^n$ of R^2*
- *Polynomial injections $U_i \to R$ such that the images are disjoint and diophantine in R.*

Then (R, L) admits diophantine storing.

Proof. Define $f : R^2 \to R$ by $f(x) = y \iff x \in U_i \wedge f_i(x) = y$.

Then we define an interpretation ϕ_f of L in $\mathrm{im}(f)$:

If P is a predicate of L of arity i, define

$$(y_1, \ldots, y_i) \in \phi_f(P) \iff \exists (x_{1_1}, x_{1_2}, \ldots, x_{i_1}, x_{i_2}) \in R^{2i} \left(\bigwedge_{j=1}^{i} y_j = f(x_{j_1}, x_{j_2}) \right)$$
$$\wedge ((x_{1_1}, x_{1_2}), \ldots (x_{i_1}, x_{i_2})) \in \phi^2(P)$$

As R admits $\wedge$, the above is diophantine up to ϕ in (R, L, ϕ), and we have a bijection $(\mathrm{im}(f), L, \phi_f) = (R^2, L, \phi^2)$.

$\mathrm{im}(\phi)$ is Diophantine as R admits $\vee$.

□

$\mathbb{Z}$ admits diophantine storage, and surjective storage by 4 functions

Theorem ([2])**.** $(\mathbb{Z}, (0, 1, +, \cdot, =))$ *admits storage, and admits surjective storage with* 4 *functions.*

Proof. Considering the function of Cantor from $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$

$$\pi(m, n) = \frac{1}{2}(m + n)(m + n + 1) + n \tag{1}$$

and the injective polynomial $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0} : x \mapsto 2x^2 + x + 1$, the function $2\pi(f(x), f(y))$ is a storage function for $\mathbb{Z}$.

Now consider the 4 functions

$$(x, y) \mapsto 2\pi(x, y) \qquad \text{if } x \geq 0, y \geq 0 \tag{2}$$
$$(x, y) \mapsto -2\pi(-x, -y) \qquad \text{if } x \leq 0, y \leq 0 \text{ and } (x, y) \neq (0, 0) \tag{3}$$
$$(x, y) \mapsto 2\pi(-x, y) - 1 \qquad \text{if } x < 0, y > 0 \tag{4}$$
$$(x, y) \mapsto -2\pi(x, -y) + 1 \qquad \text{if } x > 0, y < 0 \tag{5}$$
$$\tag{6}$$

The images are $2\mathbb{Z}_{\geq 0}, -2\mathbb{Z}_{\geq 0}, 2\mathbb{Z}_{\geq 0} - 1, -2\mathbb{Z}_{\geq 0} + 1$, which are all diophantine in $\mathbb{Z}$ by Lagrange's four-square theorem (and as $x \neq 0 \iff x \geq 1 \vee -x \geq 1$).

□

It is an open question as to whether $\mathbb{Z}$ admits surjective storing by one funtion.

n-abc-conjecture

Define the radical as, where all $e_i > 0$,

$$\mathrm{rad}\left(\prod_i p_i^{e_i} \right) = \prod_i p_i \tag{7}$$

Conjecture (abc-conjecture, common form)**.** *Let $\epsilon > 0$. Then there exist only finitely many $a + b = c$ coprime positive integers such that*

$$c < \mathrm{rad}(abc)^{1+\epsilon} \tag{8}$$

This conjecture has a very wide variety of consequences, Wikipedia alone lists 16 such, among them Fermat's last theorem, Falting's theorem, and Vojta's conjecture (equivalent).

Another application is a lower bound for linear forms in logarithms.

Conjecture ([1] n-*abc*-conjecture). *Let $a_1, \ldots, a_n \in Z$, $n \geq 3$, s.t.*

- $\gcd(a_1, \ldots, a_n) = 1$
- $a_i + \cdots + a_n = 0$
- *no proper subsum of the above is zero*

Let

$$L(a_1, \ldots, a_n) = \frac{\log \max_i(|a_i|)}{\log \operatorname{rad}(a_1 \cdots a_n)} \tag{9}$$

$$\tag{10}$$

Then the L_n are uniformly bounded, and $\limsup L_n = 2n - 5$ as L_n ranges over tuples satisfying the hypotheses.

Note [1] proved $\limsup L_n \geq 2n - 5$

A conjectural injective bivariate polynomial

Theorem ([2], noted without proof). *Suppose n is odd and sufficiently large (or ≥ 25 up to checking), and the abc- and 4-abc conjecture hold. Then the function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by*

$$f(x, y) = x^n + 3y^n \tag{11}$$

is injective.

Suppose we have

$$x_1^n + 3x_2^n - x_3^n - 3x_4^n = 0 \tag{12}$$

Let $x_i =: X_i/Y_i$ with X_i, Y_i coprime.
Clear the hcf of numerators and denominators.
Then let D be minimal such that $(3)Dx_i^n$ are integers.
So we have the linear relation between the coprime integers Dx_i^n:

$$Dx_1^n + 3Dx_2^n - Dx_3^n - 3Dx_4^n = 0 \tag{13}$$

$$Y_i^n/3 \leq D \leq Y_1^n Y_2^n Y_3^n Y_4^n \tag{14}$$

Then the 4-term *abc*-conjecture implies for some absolute C

$$C \geq \frac{\log \max_i\{Dx_i^n\}}{\log rad(9D^4 x_1^n x_2^n x_3^n x_4^n)} \tag{15}$$

Now wlog there is at least one $|x_i| \geq 1$ as we can divide through by x_i^{2n}.

Then considering $|Dx_i^n|$, D will always clear the denomiator of x_i, so it is $\geq X_i$, and as there is some $|x_i| > 1$, it is $\geq Y_i/3$.

So with H the max height of all of these, we have (for big enough H, owing to the limit aspect of the 4-*abc*, or by replacing $3+\frac{1}{8}$ below with a larger uniform bound),

$$3 + \frac{1}{8} > \frac{n \log H - \log 3}{\log rad(Y_1^{3n} \cdots Y_4^{3n} \cdots X_1 X_2 X_3 X_4)} \tag{16}$$

$$\geq \frac{n \log H}{\log Y_1 \cdots X_4} \geq \frac{n - \log 3 / \log H}{8} \tag{17}$$

So it is injective with $n \geq 25$ on tuples of sufficient height (an effective *abc* would allow checking the lower ranges or give an absolute n which works uniformly).

Thus by the 4-*abc*-conjecture some subsum must be zero.

The only nontrivial subsum is where three sum to zero, for which the *abc*-conjecture similarly shows impossible.

Note that it was assumed n odd for obvious reasons, and $H > 1$, which illustrates the necessity of having the coefficient of 3 as opposed to 2 ($(-1)^n + 2 \cdot 1^n - 1^n = 0$) or 1 (obviously) for the subsum of three case.

Function field analogue

The function field analogue of this conjectural result is proved using Mason's analogue of the *abc*-conjecture for function fields, and its generalizations

Let $a(t), b(t), c(t)$

$$\max\{\deg(a, b, c)\} \leq \deg \operatorname{rad}(abc) - 1 \tag{18}$$

References

[1] J. Browkin and J. Brzeziński. Some remarks on the *abc*-conjecture. *Math. Comp.*, 62(206):931–939, 1994.

[2] Gunther Cornelissen. Stockage diophantien et hypothèse *abc* généralisée. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(1):3–8, 1999.