Pfaffian control of some polynomials involving the j–function and Weierstrass elliptic functions

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1 Introduction

In this paper, we obtain some new bounds on the number of zeros of polynomials in z and $j(z)$, and polynomials in z and $\wp(z)$, where $\wp$ is a Weierstrass $\wp$–function associated to a lattice of the form $\langle 1, i\tau\rangle$, where τ is real. By the argument principle, the zeros of a holomorphic function in a closed, compact, simply connected region of the complex plane are controlled by the winding number of the function on its boundary – if the function is tame on the boundary, then we will obtain control over the number of zeros within the region. For j, and $\wp$, we use Pfaffian definitions of their inverses on suitable contours, and results of Khovanskii [7] to bound the zeros.

For polynomials in z and $j(z)$, we obtain the following,

Theorem 1. *Let $P(X, Y)$ be a complex polynomial of degree at most d in either variable. Then $P(z, j(z))$ has at most $2^{68}d^{10}$ zeros in the standard fundamental domain.*

This we believe is a new result, giving a bound on the whole (non–compact) fundamental domain. It may be compared to Binyamini's result of a similar nature for Noetherian functions on relatively compact domains [3]. An application of this result here would yield a bound which depends on the size of a domain in which the zeros lie, and the size of $|j|$ on it. Note also that j is o–minimal, which gives an ineffective finiteness result. The zero–bound depends polynomially on degree, and in that sense is not too far from the truth, as an obvious lower bound is $\gg d^2$. It is known that the inverse of $j(ix)$ is real and Pfaffian on the imaginary axis, a fact which may be deduced from its expression in terms of the Gaussian hypergeometric functions, which are themselves Pfaffian on an interval, see for example [6] and [2], who make use of this in the expression of elliptic functions. We make use of a result from Ramanujan's theory of elliptic functions to alternative bases to obtain a direct expression for the inverse in terms of Gaussian hypergeometric functions. Near the cusps it approximates to the q^{-1} term in its q-expansion. Though j is a Noetherian function, it is unknown whether its real and imaginary parts are Pfaffian in the whole fundamental domain, which would directly furnish a zero–bound, so considering a contour containing the fundamental domain avoids this.

For the Weierstrass $\wp$ functions, we restrict our attention to those with square lattice, as in this case $\wp$ is real on the boundary of any fundamental domain, which improves the estimates under consideration, and obtain the following,

Theorem 2. *Suppose that $\wp(z)$ is the Weierstrass $\wp$ function associated to the lattice $\langle 1, i\tau\rangle$, where $\tau > 0$ is real, and let $P(X, Y)$ be a complex polynomial of degree d in either variable. Then $P(z, \wp(z))$ has at most $8d^2 + 14d + 5$ zeros in each fundamental domain.*

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This may be compared with the non–uniform bound $c(\wp)d^2$ (see for example [8]), which holds for all $\wp$. A similar result is that of Jones and Schmidt [6] – they give a Pfaffian definition of $\wp$ on the whole fundamental domain, which yields a uniform bound of cd^{10}, where c is an absolute constant. For non–real invariants, a similar result by our method of inferior quality could be obtained, as the need to take real and imaginary parts complicates the Pfaffian definitions involved.

2 Preliminaries

We first have two lemmas bounding the arc integral related to the argument principle, which we will apply to segments of a closed contour, the first to estimate where there is a dominant term, and the second in order to apply bounds on zeros of real functions.

Lemma 1. *Suppose that $|f(z)| > C|g(z)|$ for some $C > 1$ on the contour Γ, and $f(z), g(z)$ are non–zero on Γ. Then*

$$\left|\int_\Gamma \frac{(f+g)'(z)}{(f+g)(z)}dz\right| \le \left|\int_\Gamma \frac{f'(z)}{f(z)}dz\right| + \frac{C}{C-1}|\Gamma| \sup_{z\in\Gamma}\left\{\frac{|f'(z)||g(z)|}{|f(z)|^2} + \frac{|g'(z)|}{|f(z)|}\right\},$$

where $|\Gamma|$ is the length of Γ.

Proof. Considering the difference, we have

$$\begin{aligned}\left|\int_\Gamma \left(\frac{f'(z)}{f(z)} - \frac{f'(z)+g'(z)}{f(z)+g(z)}\right)dz\right| &= \left|\int_\Gamma \frac{f'(z)g(z)+g'(z)f(z)}{f(z)(f(z)+g(z))}dz\right| \\ &\le \int_\Gamma \frac{|f'(z)g(z)|+|g'(z)f(z)|}{|f(z)|(|f(z)|-|g(z)|)}dz \\ &\le \frac{|\Gamma|}{1-\frac{1}{C}} \sup_{z\in\Gamma}\left\{\frac{|f'(z)||g(z)|}{|f(z)|^2} + \frac{|g'(z)|}{|f(z)|}\right\}.\end{aligned}$$

□

Lemma 2.

$$\frac{1}{2\pi}\left|\int_\Gamma \frac{f'(z)}{f(z)}dz\right| \le \#\{\mathrm{I}m(f(z)) = 0 | z\in\Gamma\}/2 + 1$$

and

$$\frac{1}{2\pi}\left|\int_\Gamma \frac{f'(z)}{f(z)}dz\right| \le \#\{\mathrm{R}e(f(z)) = 0 | z\in\Gamma\}/2 + 1.$$

The following definition is less general that that of Pfaffian functions in [7], but is easier to work with (and restricted to one dimension),

Definition 1. *Let $f_1, \ldots, f_r$ be a sequence of analytic functions on the interval (a, b). Then $f_1, \ldots, f_r$ is a* Pfaffian chain *of degree α if, for $1 \le i \le r$ and $x \in (a, b)$,*

$$\frac{df_i(x)}{dx} = P(x, f_1(x), \ldots, f_i(x)),$$

and the maximum total degree of each P_i is α. A Pfaffian function *of order r and degree (α, β) is a function $P(x, f_1(x), \ldots, f_s(x))$ where P is a polynomial of total degree at most β, and $f_1(x), \ldots, f_s(x)$ are members of a Pfaffian chain of order r and degree α.*

We make use of the following bound on the number of zeros of a Pfaffian function,

Theorem 3 ([7])**.** *Let f be a Pfaffian function of order r and degree (α, β) on the open interval (a, b). Then the number of zeros of f in (a, b) is at most*

$$2^{r(r-1)/2}\beta(\alpha+\beta)^r.$$

3 Polynomials in z and $j(z)$

Here we make use of an expression for the inverse of j in terms of the Gaussian hypergeometric function, which is given, for $|z| < 1$, by

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(a)_n = \prod_{k=0}^{n-1}(a+k)$ is the rising factorial, and c not a non–positive integer. Gauss determined 15 relations between those ${}_2F_1$ whose parameters differ by integers. We make use of the following two relations, where we have suppressed the parameters of the function, and use the notation $F(c\pm) = {}_2F_1(a, b, c\pm 1; z)$,

Theorem 4 (Gauss, [5], Art. 11, Eq. 16; DLMF [1], Eq. 15.5.21)**.**

$$\begin{aligned} z\frac{dF}{dz} &= (c-1)F(c-) - F) \\ &= z\frac{(c-a)(c-b)F(c+) + c(a+b-c)F}{c(1-z)}. \end{aligned}$$

Now these allow us to construct a Pfaffian chain which j^{-1} will be defined over on the imaginary axis.

Lemma 3. *The sequence of functions*

$$\frac{1}{x}, \qquad \frac{1}{1-x}, \qquad \frac{F}{F(c+)}, \qquad F(c+), \qquad F, \qquad \frac{1}{F}$$

is a Pfaffian chain of degree 3.

Proof. We first note that $\frac{1}{x}$ and $\frac{1}{1-x}$ are Pfaffian of order 1 and degree $(2, 1)$. By Gauss' contiguous relations, we have

$$\begin{aligned} x\frac{dF(c+)}{dx} &= c(F - F(c+)) \\ \frac{dF}{dx} &= \frac{(c-a)(c-b)F(c+) + c(a+b-c)F}{c(1-x)}. \end{aligned}$$

Consider

$$\begin{aligned} \frac{d}{dx}\frac{F}{F(c+)} &= \frac{(c-a)(c-b)F(c+) + c(a+b-c)F}{c(1-x)F(c+)} - \frac{c(F - F(c+))F}{xF(c+)^2} \\ &= \frac{(c-a)(c-b)}{c(1-x)} + \left(\frac{a+b-c}{c(1-x)} - \frac{c-1}{x}\right)\frac{F}{F(c+)} - \frac{c}{x}\left(\frac{F}{F(c+)}\right)^2. \end{aligned}$$

Hence $\frac{F}{F(c+)}$ is a Pfaffian function of order 3 and degree $(2,1)$. Now consider

$$\frac{dF(c+)}{dx} = \frac{c(F - F(c+))}{x} = \frac{c}{x}\left(F(c+)\frac{F}{F(c+)} - F(c+)\right),$$

so $F(c+)$ is a Pfaffian function of order 4 and degree $(3,1)$. Hence $F = F(c+)\frac{F}{F(c+)}$ is a Pfaffian function of order 5 and degree $(3,2)$. Further, $1/F$ is Pfaffian of order 6 and degree $(3,1)$. $\square$

We make use of the following consequence, which is clear considering the above argument for ${}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{1}{2}+\frac{y}{2}\right)$ and ${}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{1}{2}-\frac{y}{2}\right)$.

Lemma 4. $\frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{1}{2}+\frac{y}{2}\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{1}{2}-\frac{y}{2}\right)}$ *is Pfaffian function of order* 9 *and degree* $(2,1)$ *on* $(0,1)$.

Now we express the inverse of j on the imaginary axis in terms of the hypergeometric functions by way of the following theorem, an inversion formula from the theory of elliptic functions to alternative bases – in this case the sextic theory.

Theorem 5 (Theorem 4.10, $r=1$, [4])**.** *Let q be a real number in the interval* $0 < q < 1$. *Then*

$$q = \exp\left(-2\pi\frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,1-x\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,x\right)}\right),$$

where, letting P,Q,R be Ramanujan's Eisenstein series,

$$x(1-x) = \frac{Q(q)^3 - R(q)^2}{4Q(q)^3}.$$

Letting $q = e^{2\pi i\tau}$, and τ be purely imaginary, we take

$$\alpha = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1728}{j(\tau)}},$$

which satisfies

$$\alpha(1-\alpha) = \frac{1728}{4j(\tau)} = \frac{Q(q(\tau))^3 - R(q(\tau))^2}{4Q(q(\tau))^3},$$

so that by Theorem 5,

$$\tau = i\frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{1728}{j(\tau)}}\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{1728}{j(\tau)}}\right)}.$$

So for $x \geq 1728$, letting $J(x) = j(ix)$,

$$J^{-1}(x) = \frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{1728}{x}}\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{1728}{x}}\right)}.$$

Proof of Theorem 1. We apply the argument principle to a truncated fundamental domain together with its copies or half–copies under $\mathrm{SL}_2(\mathbb{Z})$ indicated by Figure 1. First consider a contour Γ within the one indicated, containing all zeros within the original contour, such that $P(z,j(z))$ is non-zero on this contour. Let $0 < \epsilon < \min_{z\in\Gamma}|P(z,j(z))|$. Then by Rouché's theorem, the

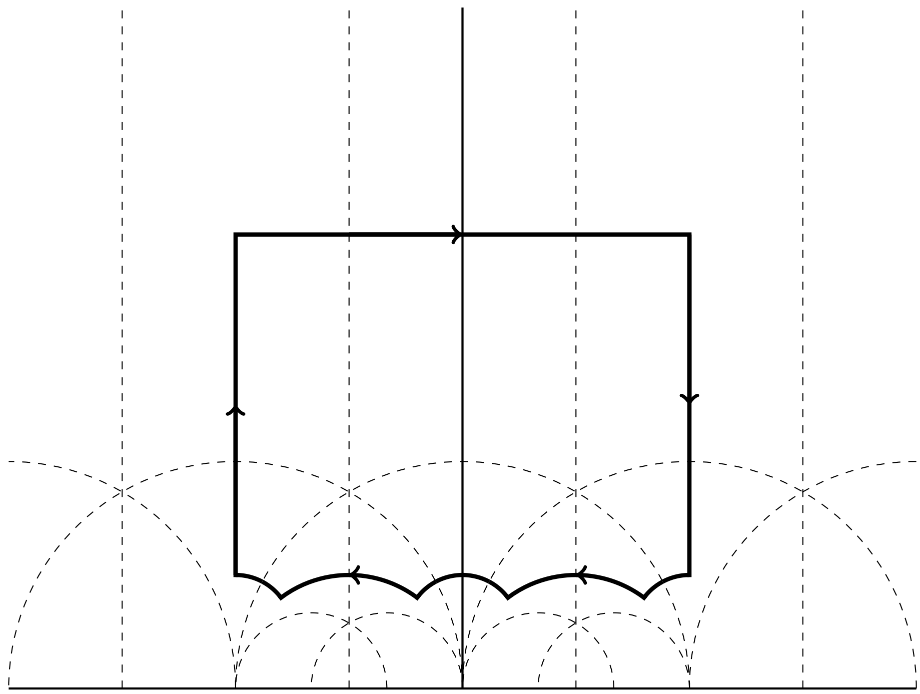

FIGURE 1

number of zeros of $P(z, j(z)) + \epsilon e^{i\theta}$ within Γ is equal to that of $P(z, j(z))$, for any θ. Choose θ so that $P_\epsilon(z, j(z)) := P(z, j(z)) + \epsilon e^{i\theta}$ is non–zero on the original contour – this is possible by discreteness of zeros of analytic functions.

We now bound the winding number of $P_\epsilon(z, j(z))$ on this boundary. The contour is chosen to be dominated by the q^{-1} term of j near the cusps. Let $j(it) = J(t)$. We will refer to elements of $\mathrm{SL}_2(\mathbb{Z})$ by g, with entries

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$$

acting on z by

$$g(z) = \frac{\tilde{a}z + \tilde{b}}{\tilde{c}z + \tilde{d}}.$$

The contour is composed of lines or curves near the cusps, and images of the imaginary axis under the action of some g.

Copies of the imaginary axis $i\mathbb{R}_{\geq 1}$: Here we have $\mathrm{Im}(P(g(it), j(g(it)))) = \mathrm{Im}\left(P\left(\frac{\tilde{a}it+\tilde{b}}{\tilde{c}it+\tilde{d}}, j(it)\right)\right) = \mathrm{Im}\left(P\left(\frac{(\tilde{a}it+\tilde{b})(-\tilde{c}it+\tilde{d})}{\tilde{c}^2t^2+\tilde{d}^2}, j(it)\right)\right) = \left(\frac{1}{\tilde{c}^2t^2+\tilde{d}^2}\right)^d Q(t, J(t))$ for some real Q, as $j(it)$ is real. Q is of degree $\leq 2d$, and has the same number of zeros as $\mathrm{Im}(P(g(it), j(g(it))))$.

As $J(t)$ is 1–1 from $(1, \infty)$ to $(1728, \infty)$, letting $x = J(t)$, the number of zeros of $Q(t, J(t))$ is equal to that of $Q(J^{-1}(x), x)$ on $x > 1728$, and as $\sqrt{1 - \frac{1728}{x}}$ is 1–1 on $x > 1728$, letting

$y = \sqrt{1 - \frac{1728}{x}}$, the number of zeros of $Q(J^{-1}(x), x)$ is equal to that of

$$Q\left(J^{-1}\left(\frac{1728}{1-y^2}\right), \frac{1728}{1-y^2}\right) = \left(\frac{1}{1-y^2}\right)^{2d} R\left(J^{-1}\left(\frac{1728}{1-y^2}\right), y\right)$$

on $(0,1)$, where R is of degree $\leq 4d$. Now using the expression for J^{-1} in terms of the hypergeometric series ${}_2F_1$, we have that the number of zeros of $\mathrm{Im}(P(g(it), j(g(it))))$ is bounded by that of

$$R\left(\frac{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \frac{1}{2} + \frac{y}{2}\right)}{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \frac{1}{2} - \frac{y}{2}\right)}, y\right)$$

for $y \in (0,1)$. Now by Lemma 4, this is a Pfaffian function of order 9 and degree $(3, 4d)$. By Theorem 3, the number of zeros is then bounded by $2^{64}d^{10}$, so by Lemma 2, on this copy of the imaginary axis, we have

$$\frac{1}{2\pi}\left|\int_\gamma \frac{d}{dz}(P(z, j(z)))\frac{1}{P(z, j(z))}dz\right| \leq 2^{64}d^{10}.$$

Copies of $\mathrm{Im}(z) = Y$: For these sections of the boundary, we have

$$P(g(x+iY), j(g(x+iY))) = \left(\frac{1}{\tilde{c}(x+iY)+\tilde{d}}\right)^d Q(x+iY, j(x+iY)),$$

so that

$$\int_\gamma \frac{\frac{d}{dz}(P(g(z), j(z)))}{P(g(z), j(z))}dz = \int_\gamma \frac{\frac{d}{dz}(Q(z, j(z)))}{Q(z, j(z))} - \frac{d\tilde{c}}{\tilde{c}z+\tilde{d}}dz.$$

For sufficiently large Y, $\frac{d\tilde{c}}{\tilde{c}z+\tilde{d}} < 0.01$, so we consider the remaining term in Q.

Letting $j(z)^l h(z)$ be the term with the largest power of j occurring in $Q(z, j(z))$, where $h(z)$ is its coefficient over the polynomials in z, let the degree of h be n. For sufficiently large Y, $j(x+iY)(iY)^n$ is the dominant term in $Q(x+iY, j(x+iY))$, i.e. $|j(x+iY)(iY)^n| > 2|Q(x+iY, j(x+iY)) - j(x+iY)(iY)^n|$. In the same way the $e^{-2\pi i(x+iY)}$ term in the q-expansion of j dominates $j(x+iY)$ when Y is large, and so as $Y \to \infty$,

$$\frac{\left|Q(x+iY, j(x+iY)) - e^{-2l\pi i(x+iY)}(iY)^n\right|}{\left|e^{-2l\pi i(x+iY)}(iY)^n\right|} \to 0,$$

and the same holds for the derivative of the numerator. So taking $f(z) = (iY)^n e^{-2l\pi iz}$ and $g(z) = Q(z, j(z)) - f(z)$, for sufficiently large Y, we may take $C = 2$ in Lemma 1, and obtain the bound

$$\begin{aligned}\left|\int_\gamma \frac{\frac{d}{dz}(Q(z, j(z)))}{Q(z, j(z))}dz\right| &\leq \left|\int_\gamma \frac{\frac{d}{dz}(e^{-2l\pi iz}(iY)^n)}{e^{-2l\pi iz}(iY)^n}dz\right| + 2\sup_{z\in\gamma}\left\{\frac{|f'(z)||g(z)|}{|f(z)|^2} + \frac{|g'(z)|}{|f(z)|}\right\} \\ &\leq 2\pi l + 0.01 \\ &\leq 2\pi d + 0.01.\end{aligned}$$

We also take Y sufficiently large to ensure all zeros in the fundamental domain (and its copies) are interior to the contour, which is possible by virtue of the dominating term in $P(z, j(z))$ at the various cusps (or by the o–minimality of j implying there are only finitely many). Finally, the integral over the entire contour is bounded in absolute value by $8 \cdot 2^{64}d^{10} + 10d + 0.2 \leq 2^{68}d^{10}$, as there are 8 copies of the imaginary axis, and 10 copies or half–copies of the line $(-1/2 + iY, 1/2 + iY)$.

□

4 Polynomials in z and the Weierstrass $\wp$–function

Here we make use of the following theorem of Khovanskii,

Theorem 6 (§2.3, Theorem 2, [7]). *Let $G : R^{n+1} \to R^1$ be a smooth function with nondegenerate level set M^n. Let $F : R^{n+1} \to R^n$ be a smooth proper map, and $\tilde{F} : M^n \to R^n$ its restriction to M^n. Let, further, $\hat{J}$ be any smooth function on R^n that coincides on M^n with the Jacobian J of the map $(F, G) : R^{n+1} \to R^n \times R^1$. Under these conditions the following holds: the maximum number of nondegenerate preimages of any point in the range of of the map $\tilde{F} : M^n \to R^n$ is bounded by that of the map $(F, \hat{J}) : R^{n+1} \to R^n \times R^1$.*

We apply this to a polynomial in x and $\wp^{-1}(x)$, where $\wp$ is a Weierstrass $\wp$–function, and use the argument principle to bound the zeros of $P(z, \wp(z))$ in its fundamental domain. We consider $\wp$ with lattices of the form $\langle 1, i\tau \rangle$, $\tau > 0$ real. The bound on the number of zeros follows in a similar way to §2.3 Theorem 1 of [7] – we proceed in this manner to give a better bound than simply applying Theorem 3.

First we have the differential equation satisfied by $\wp$,

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

and $\tilde{\wp}(x) := \wp(ix)$ satisfies the equation

$$\tilde{\wp}'(x)^2 = -4\wp(z)^3 + g_2\wp(z) + g_3$$

where g_2, g_3 are the Weierstrass invariants of $\wp$, which are real as $\wp$ is associated to the lattice $\langle 1, i\tau \rangle$. For the derivatives of the inverses of $\wp(x)$ and $\tilde{\wp}(x)$, we have the expressions

$$\frac{d}{dx}\wp^{-1}(x) = \frac{1}{\wp'(\wp^{-1}(x))} = \frac{1}{\sqrt{4x^3 - g_2x - g_3}},$$
$$\frac{d}{dx}\tilde{\wp}^{-1}(x) = \frac{1}{\tilde{\wp}'(\tilde{\wp}^{-1}(x))} = \frac{1}{\sqrt{-4x^3 + g_2x + g_3}}$$

where the sign of the square root depends on the branch of the inverse of $\wp$ or $\tilde{\wp}$ which is under consideration. Note that both expressions will be real in the domains under consideration.

Proposition 1. *Let $\wp$ have lattice $\langle 1, i\tau \rangle$, where $\tau > 0$ is real, and $P(X, Y)$ be a complex polynomial of total degree at most d. Then the number of zeros of $\mathrm{Im}(P(z, \wp(z)))$ on the open line $(\beta, \beta + \gamma)$, $\gamma \in \{1, i\tau\}$ between two adjacent poles of $\wp$ is bounded by $4d^2 + 6d + 1$.*

Proof. By periodicity, $\wp(z) = \wp(z - \beta)$, so we may take a transformation of z to consider the lines $(0, 1)$, and $(0, i\tau)$. As $\wp$ has lattice $\langle 1, i\tau \rangle$, and τ is real, it is real on these lines, and $\mathrm{Im}(P(z, \wp(z))) = Q(x, \wp(\gamma x))$, $\gamma \in \{1, i\}$ for some real polynomial $Q(X, Y)$ of degree at most d. The argument proceeds identically for either line, so we consider $(0, 1)$.

Let $x_0 \in (0, 1)$ be such that $\wp'(x_0) = 0$. Then $\wp(x)$ is 1–1 on the interval $(0, x_0)$, so the number of zeros of $Q(x, \wp(x))$ is equal to that of $Q(\wp^{-1}(x), x)$ on the interval $(\wp(x_0), \infty)$. Considering the system

$$Q(u, x) = 0$$
$$u - \wp^{-1}(x) = 0$$

we have, letting J be the Jacobian of the system $(Q, u - \wp^{-1}(x))$, by Theorem 6, that the number of nondegenerate solutions the system is bounded by an upper bound of the number of

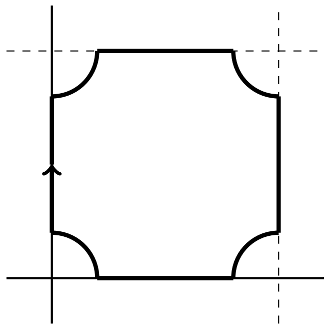

FIGURE 2

nondegenerate preimages of any point in the range of the system

$$\begin{aligned} &Q(u,x)\\ &J(u,x). \end{aligned}$$

Letting $Q_X(X,Y) = \frac{\partial}{\partial X}Q(X,Y)$, and similarly for $Q_Y(X,Y)$, $J(x,u)$ is given by

$$\frac{Q_X(u,x)}{\sqrt{4x^3 - g_2x - g_3}} + Q_Y(u,x).$$

Taking some point (a,b) in the range of the system (Q,J), we bound the number of nondegenerate preimages. If $J(x,u) = b$, then

$$J(u,x) = \frac{Q_X(u,x)}{\sqrt{4x^3 - g_2x - g_3}} + Q_Y(u,x) = b,$$

and this holds iff

$$Q_X(u,x) + (Q_Y(u,x) - b)\sqrt{4x^3 - g_2x - g_3} = 0,$$

and if this holds, then

$$\begin{aligned} &(Q_X(u,x) + (Q_Y(u,x) - b)\sqrt{4x^3 - g_2x - g_3})\\ &\quad \cdot (Q_X(u,x) - (Q_Y(u,x) - b)\sqrt{4x^3 - g_2x - g_3})\\ &\quad = Q_X(u,x)^2 - (Q_Y(u,x) - b)^2(4x^3 - g_2x - g_3) = 0 \end{aligned}$$

holds, so that the number of preimages of (a,b) is bounded by the number of nondegenerate solutions of

$$\begin{aligned} Q(u,x) - a &= 0\\ Q_X(u,x)^2 - (Q_Y(u,x) - b)^2(4x^3 - g_2x - g_3) &= 0, \end{aligned}$$

which by Bézout's theorem is bounded by $2d^2 + 3d$. The interval $(x_0, 1)$ is similar.

□

Proof of Theorem 2. Let $P(z) := P(z, \wp(z))$. We first take a box B within the fundamental domain such that all zeros of P in the interior of the fundamental domain lie within B. Next, define ϵ, θ such that $0 < \epsilon < \min_{\partial B}|P|$, and $P_\epsilon(z) := P(z) + \epsilon e^{i\theta} \neq 0$ for $z \in \partial\mathcal{F}$. By Rouché's theorem, $P_\epsilon(z)$ has the same number of zeros in B counting multiplicity as $P(z)$.

We now bound the number of zeros of P_ϵ within the contour Γ given by the truncations of the lines on the boundary of $\mathcal{F}$ united with interior quarter–circles about the poles of $\wp(z)$, as indicated in Figure 2 – the common radii of these circles is taken so that there are no zeros of P_ϵ in a disc of this radius about the poles, and such that they do not intersect B. The radii will be taken sufficiently small subject to this. As B lies in the interior of Γ, a bound upon the number of zeros of P_ϵ within Γ bounds the number within B, and so bounds that of $P(z, \wp(z))$ within the fundamental domain.

By the argument principle, the number of zeros of P_ϵ within Γ is equal to

$$\frac{1}{2\pi i}\int_\Gamma \frac{P'_\epsilon(z)}{P_\epsilon(z)}dz,$$

which we now estimate. As $\wp(z)$ is real on the lines of Γ, considering for example the line $z = ix$, for x real,

$$\mathrm{Re}(P_\epsilon(z)) = \mathrm{Re}(P(ix, \wp(ix))) + \epsilon\cos(\theta) = Q(x, \wp(ix)).$$

The number of zeros of which is, by Proposition [1], bounded by $4d^2 + 6d + 1$, and holds for any radius of quarter-circles. The other lines are similar, and so our bound for the integral over all of the lines is $8d^2 + 12d + 6$ by Lemma 2.

On the quarter–circles, we may expand P_ϵ into its Laurent series – given a sufficiently small radius δ, the term of smallest power will dominate. We let γ be the path $p + \delta e^{i\xi}$, where ξ ranges over the appropriate interval of length $\pi/2$ in $[0, 2\pi]$, for the particular pole p, to be the interior quarter–circle. This path has length $\delta\pi/4$. Writing

$$P_\epsilon(z) = a_k(z-p)^k + \sum_{n=k+1}^{\infty} a_n(z-p)^n,$$

we have, as the radius $\delta \to 0$,

$$\frac{P_\epsilon(z) - a_k(z-p)^k}{a_k(z-p)^k} \to 0,$$
$$\frac{\frac{d}{dz}(P_\epsilon(z) - a_k(z-p)^k)}{a_k(z-p)^k} \to \frac{(k+1)a_{k+1}}{a_k}.$$

So for sufficiently small δ, we have, letting $f(z) = a_k(z-p)^k$, and $g(z) = P_\epsilon(z) - f(z)$, by Lemma 1, with $C = 2$,

$$\begin{aligned}\frac{1}{2\pi}\left|\int_\gamma \frac{P'_\epsilon(z)}{P_\epsilon(z)}dz\right| &\leq \frac{1}{2\pi}\left|\int_\gamma \frac{k}{z-p}dz\right| + \frac{\delta}{2}\left(\frac{(k+1)a_{k+1}}{a_k} + 0.1\right) \\ &\leq |k|/4 + 0.1 \\ &\leq d/2 + 0.1,\end{aligned}$$

where the last inequality follows from the fact that $\wp$ has poles of order 2, so $-2d \leq k \leq d$. As there are 4 quarter–circles about the poles of $\wp$, and 4 line segments, the absolute value of the whole integral is bounded by $8d^2 + 14d + 7$. □

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