Inversion of the j–function and testing complex multiplication

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1 Introduction

In this paper we develop an algorithm to invert the j–function in quasilinear time, and give an application to testing whether an elliptic curve has complex multiplication.

It is known that the inverse of j is equal to a ratio of two Gaussian hypergeometric functions, though we do not believe any analysis has been made of the running time or required precision to invert j by this method. Our method is similar to that of [10], which makes use of addition formulae between Jacobi's theta functions in order to reduce the argument to a fixed compact set – here we repeatedly make use of the modular polynomial

$$\begin{aligned}\Phi_2(X,Y) = X^3 + Y^3 - X^2Y^2 + 1488X^2Y + 1488XY^2 - 162000X^2 - 162000Y^2 \\ +40773375XY + 8748000000X + 8748000000Y - 157464000000000,\end{aligned}$$

which has the property that the roots of $\Phi_2(j(\tau), z)$ are $j(2\tau)$, $j\left(\frac{\tau}{2}\right)$, and $j\left(\frac{\tau+1}{2}\right)$, to either compute $j(2^k\tau)$, the logarithm of which is a close approximation to $-2^{k+1}\pi\tau$, or to manipulate the argument of j into a compact set to which Newton's method may be applied.

We define the *regulated precision* of an approximation $\tilde{\alpha}$ to α to be

$$\frac{|\alpha - \tilde{\alpha}|}{\max\{1, |\alpha|\}},$$

and denote by $M(P)$ the computational complexity of multiplication of two P–bit integers, which by a recent result of Harvey and Hoeven [8] may be taken to be $O(P \log P)$. We obtain the following,

Theorem 1. *Suppose that $\tilde{j}$ is an approximation to $j(\tau)$, $\tau \in \mathcal{F}$, of regulated precision 2^{-P}, with $P \geq 400$. Let $Q = P/6$ if $|\tau - i| \leq 2^{-30}$ or $\left|\tau - \frac{\pm 1 + i\sqrt{3}}{2}\right| \leq 2^{-30}$, and $Q = P - \max\{11 \log P, 100\}$ otherwise. Then we may obtain an approximation to τ of relative precision 2^{-Q} in time*

$$O(M(P) \log(P)^2).$$

The j–function has two ramification points in its fundamental domain, which entails the loss of precision in its inversion when j is close to 0 or 1728. We apply this algorithm to test for complex multiplication of elliptic curves – given an approximation to the j–invariant of an elliptic curve and a bound upon its height and degree, we may invert it and determine if the inverse is a quadratic irrational, determining also the discriminant,

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Theorem 2. *Suppose that $j(\tau)$ is the j–invariant of an elliptic curve E, with $j(\tau)$ of degree bounded by d and height bounded by $H \geq e^e$. Then it may be determined from d, H, and an approximation to j of regulated precision $2^{-300d^2 \log H(\log d + \log\log H)^2 - 200}$ whether E has complex multiplication, and if so the associated discriminant, in time, letting $T = d^2 \log H(\log d + \log\log H)^2$,*

$$O(M(T)(\log T)^2).$$

Previous methods include that of [1], based on reduction of elliptic curves at primes, which has an unconditional running time of $O(H^{cd})$, and assuming the Generalized Riemann Hypothesis a running time of $O(d^2(\log H)^2)$, and two tests of [6], comprising a deterministic algorithm based on Galois representations associated to torsion points, with running time $O(d^{c_1}(\log H)^{c_2})$, with an ineffective implicit constant, and a probabilistic algorithm, also of polynomial running time.

We note that one may also apply our algorithm for the inversion of j to detecting isogenies between two elliptic curves of running time $O(N \log N \log\log N)$, where the degree of the isogeny is bounded by N, though our implicit constant is ineffective as explicit bounds on the coefficients of modular polynomials $\Phi_N(X, Y)$ for composite indices are not currently available.

2 Preliminaries

We will denote by $\mathcal{F}$ the usual fundamental domain of $j(z)$, $\left\{z \mid -\frac{1}{2} < \mathrm{Re}(z) \leq \frac{1}{2}, |z| > 1\right\} \cup \{z \mid |z| = 1, \mathrm{Re}(z) \geq 0\}$. Throughout we will make use of the following results,

Lemma 1 (Lemma 1 of [3]). *If $\tau \in \mathcal{F}$*

$$\left|j(\tau) - e^{-2\pi i\tau}\right| \leq 2079.$$

Theorem 3 (Kantorovich, [9]). *Let $F : S(x_0, R) \subset X \to Y$ have a continuous Fréchet derivative in $\overline{S(x_0, r)}$. Moreover, let* (i) *the linear operation $\Gamma_0 = [F'(x_0)]^{-1}$ exist;* (ii) *$\|\Gamma_0 F(x_0)\| \leq \eta$;* (iii) *$\|\Gamma_0 F''(x)\| \leq K$ $(x \in \overline{S(x_0, r)})$. Now, if*

$$h = K\eta \leq \frac{1}{2}$$

and

$$r \geq \frac{1 - \sqrt{1 - 2h}}{h}\eta,$$

then $F(x) = 0$ will have a solution x^* to which the Newton method is convergent. Here,

$$\|x^* - x_0\| \leq r_0.$$

Furthermore, if for $h < \frac{1}{2}$,

$$r < r_1 = \frac{1 + \sqrt{1 - 2h}}{h}\eta,$$

or for $h = \frac{1}{2}$

$$r \leq r_1,$$

the solution x^* will be unique in the sphere $\overline{S(x_0, r)}$. The speed of convergence is characterized by the inequality

$$\|x^* - x_k\| \leq \frac{1}{2^k}(2h)^{2^k}\frac{\eta}{h}$$

for $k = 0, 1, 2, \ldots$.

We note that the condition

$$r \geq \frac{1 - \sqrt{1-2h}}{h}\eta$$

may be replaced with

$$r \geq 2\eta.$$

3 Inversion of $j(z)$

Firstly, if $|j| \leq 2^{-P/2}$ or $|j - 1728| \leq 2^{-P/3}$, we return $\tau = \frac{1+i\sqrt{3}}{2}$ or $\tau = i$ respectively, and otherwise, we split the fundamental domain of j into 4 sections – $\mathrm{Im}(\tau) \geq 3$, $|\tau - i| \leq 2^{-31}$, $\left|\tau - \frac{1+\sqrt{3}}{2}\right| \leq 2^{-31}$, $\left|\tau - \frac{-1+\sqrt{3}}{2}\right| \leq 2^{-31}$ and the remaining compact subset of the fundamental domain. We may determine τ with sufficient precision by taking a low precision inverse via the following expression for j^{-1} in terms of Gaussian hypergeometric functions – with α a solution to

$$j(\tau) = \frac{1728}{4\alpha(1-\alpha)},$$

τ is equal to either

$$i\frac{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, 1-\alpha\right)}{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \alpha\right)}.$$

or the negative of its inverse. If j is sufficiently large, or close to 0 or 1728, then we do not need to evaluate this in order to determine which section of the fundamental domain τ lies in, so we need only compute the above to a fixed precision in a compact set, at points which it is easily shown are bounded away from zeros of ${}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, z\right)$, so this takes only constant time.

3.1 Large j

Throughout this section we assume $\mathrm{Im}(\tau) \geq 3$, and will repeatedly make use of the consequent fact that $|j(\tau)| \geq 10^8$. We will make use of the modular polynomial $\Phi_2(X, Y)$ to obtain an approximation to $j(2\tau)$, and repeat the process until we have an approximation to $j(2^k\tau)$, where k is sufficiently large, at which point taking the logarithm of $j(2^k\tau)$ gives a good approximation to $-2^{k+1}\pi\tau$, owing to the q–series of j.

Proposition 1. *Let $j(\tau) = j$, and suppose that $\mathrm{Im}(\tau) \geq 3$ and $\tilde{j}$ is an approximation to j of relative precision at least 2^{-P}, with $P \geq 300$. Then the largest root, in absolute value, of $\Phi_2(\tilde{j}, z)$ is an approximation to $j(2\tau)$ of relative precision at least 2^{-P+2}.*

Proof. Let $f(z) = \Phi_2(j, z)$ and $g(z) = \Phi_2(\tilde{j}, z)$. We first bound the coefficients of $f(z) - g(z)$. For the coefficient of z^2, we have, with $\tilde{j} = j + \delta$,

$$\begin{aligned} \left|-j^2 + 1488j - 162000 - (-(j+\delta)^2 + 1488(j+\delta) - 162000)\right| &= |2\delta j + \delta^2 + 1488\delta| \\ &\leq 2.1|\delta||j|, \end{aligned}$$

as $|\delta| \leq 2^{-300}|j|$ and $|j| \geq 10^8$. For the coefficient of z, we have

$$\begin{aligned} &\left|1488j^2 + 40773375j + 8748000000 - (1488(j+\delta)^2 + 40773375(j+\delta) + 8748000000)\right| \\ &\quad = |2976\delta j + 1488\delta^2 + 40773375\delta| \\ &\quad \leq 3210|\delta||j|. \end{aligned}$$

For the constant coefficient, we have

$$\begin{aligned}&\left|3\delta j^2+3\delta^2 j+\delta^3-16200\delta j-16200\delta^2+8748000000\delta\right|\\&\quad\leq 3.4|\delta||j|^2.\end{aligned}$$

We now bound the values of $g(j(\tau'))$ for $\tau'\in\left\{2\tau,\frac{\tau}{2},\frac{\tau+1}{2}\right\}$. For $j(2\tau)$, as $|j(2\tau)|\leq 1.02|j|^2$,

$$\begin{aligned}|g(j(2\tau))|&=|g(j(2\tau))-f(j(2\tau))|\\&\leq 2.1|\delta||j|(1.02|j|^2)^2+3210|\delta||j|\cdot 1.02|j|^2+3.4|\delta||j|^2\\&\leq 2.2|\delta||j|^5.\end{aligned}$$

For $g(j(\tau'))$, $\tau'\in\left\{\frac{\tau}{2},\frac{\tau+1}{2}\right\}$, firstly, the absolute values of $j\left(\frac{\tau}{2}\right)$ and $j\left(\frac{\tau+1}{2}\right)$ are bounded above and below by $e^{2\pi\mathrm{Im}(\tau)/2}\pm 2079$, and the absolute value of j is bounded above and below by $e^{2\pi\mathrm{Im}(\tau)}\pm 2079$. As $\mathrm{Im}(\tau)\geq 3$, these yield

$$0.83|j|^{1/2}\leq\left|j\left(\frac{\tau+1}{2}\right)\right|,\left|j\left(\frac{\tau}{2}\right)\right|\leq 1.17|j|^{1/2},\tag{1}$$

so

$$\begin{aligned}|g(j(\tau'))|&=|g(j(\tau'))-f(j(\tau'))|\\&\leq 2.1|\delta||j|(1.17|j|^{1/2})^2+3210|\delta||j|(1.17|j|^{1/2})+3.4|\delta||j|^2\\&\leq 20|\delta||j|^2.\end{aligned}$$

Let β_0,β_1,β_2 be the roots of $g(z)$. Then for β_i the closest root of g to $j(2\tau)$,

$$|j(2\tau)-\beta_i|\leq\left(2.2|\delta||j|^5\right)^{1/3}\leq\left(2.2\cdot 2^{-100}|j|^6\right)^{1/3}\leq 10^{-9}|j|^2,$$

and we let β_0 be the closest root of g to $j(2\tau)$. As $0.98|j|^2\leq j(2\tau)\leq 1.02|j|^2$, $0.97|j|^2\leq|\beta_0|\leq 1.03|j|^2$. For $\tau'=\frac{\tau}{2}$, with β_i the closest root of g to $j(\tau')$,

$$|j(\tau')-\beta_i|\leq\left(20|\delta||j|^2\right)^{1/3}\leq\left(20\cdot 2^{-300}|j|^3\right)^{1/3}\leq 10^{-29}|j|,$$

and in particular,

$$|\beta_i|\leq 10^{-29}|j|+1.17|j|^{1/2}<0.97|j|^2\leq|\beta_0|,\tag{2}$$

so $\beta_i\neq\beta_0$. Let this root of g be β_1. Now we may improve the bound on $|j(\tau')-\beta_1|$,

$$|j(\tau')-\beta_1|^2\leq\frac{20|\delta||j|^2}{|j(\tau')-\beta_0|}\leq\frac{20|\delta||j|^2}{0.98|j|^2-1.17|j|^{1/2}}\leq 30\cdot 2^{-300}|j|,$$

so

$$|j(\tau')-\beta_1|\leq 10^{-88}|j|^{1/2},$$

and consequently

$$0.82|j|^{1/2}\leq|\beta_1|\leq 1.18|j|^{1/2}.$$

We now bound β_2. By our bound on the difference between the constant coefficients of f and g, the constant coefficient $-\beta_0\beta_1\beta_2$ of g is bounded in absolute value by

$$\left|j(2\tau)j\left(\frac{\tau}{2}\right)j\left(\frac{\tau+1}{2}\right)\right|+3.4\delta|j|^2\leq 1.4|j|^3+0.01|j|^3\leq 1.5|j|^3,$$

so that

$$|\beta_2| \le \frac{1.5|j|^3}{0.97|j|^2 \cdot 0.82|j|^{1/2}} \le 2|j|^{1/2}.$$

Returning to $g(j(2\tau))$, we now bound below the terms $|j(2\tau)-\beta_i|$ for $i=1,2$ in order to improve our inequality for $|j(2\tau)-\beta_0|$. We now have, for $i=1,2$,

$$\begin{aligned} |j(2\tau)-\beta_i| &\ge |j(2\tau)| - 2|j|^{1/2} \\ &\ge 0.97|j|^2. \end{aligned}$$

This now improves our bound on the difference of β_0 to $j(2\tau)$,

$$\begin{aligned} |j(2\tau)-\beta_0| &\le \frac{2.2\delta|j|^5}{|j(2\tau)-\beta_1||j(2\tau)-\beta_2|} \\ &\le \frac{2.2\delta|j|^5}{(0.97|j|^2)^2} \\ &\le 2.4\delta|j|. \end{aligned}$$

So the relative precision of β_0 as an approximation to $j(2\tau)$ is bounded by

$$\frac{|j(2\tau)-\beta_0|}{|j(2\tau)|} \le \frac{2.4|\delta||j|}{0.97|j|^2} \le 2.5\frac{|\delta|}{|j|} \le 2^{-P+2}.$$

□

Proposition 2. *If* $\mathrm{Im}(\tau) \ge 3$ *and* $\tilde{j}$ *is an approximation to* $j(\tau)$ *of relative precision* 2^{-P}*, where* P *is at least* 300*, applying Newton's method to* $\Phi_2(\tilde{j}, z)$*, with starting point*

$$\tilde{j}^2 - 2\cdot 744\tilde{j} - 2\cdot 196884 + 744^2 + 744,$$

will obtain an approximation to $j(2\tau)$ *of relative precision* 2^{-P+3} *after at most* $[2\log P + \log\log|j|]$ *steps of Newton iteration.*

Proof. We first give a rough approximation to β_0. Writing

$$j(\tau) = e^{-2\pi i\tau} + 744 + 196884e^{2\pi i\tau} + f(\tau),$$

we have

$$\begin{aligned} &j(\tau)^2 - 2\cdot 744 j(\tau) + 2\cdot 19688 + 2\cdot 744^2 + 744 \\ &\qquad = e^{-4\pi i\tau} + 744 + 196884^2 e^{4\pi i\tau} + 2(196884e^{2\pi i\tau} + e^{-2\pi i\tau})f(\tau) + f(\tau)^2 \end{aligned}$$

so that, as $\mathrm{Im}(\tau) \ge 3$, f is maximized when $\mathrm{Re}(\tau) = 0$, and $f(\mathrm{Im}(\tau))$ is decreasing in $\mathrm{Im}(\tau)$,

$$\begin{aligned} &|j(2\tau) - (j(\tau)^2 - 744j(\tau) + 19688 - 744^2 + 744)| \\ &\quad \le (196884 + 196884^2)e^{-4\pi \mathrm{Im}(\tau)} + |f(\mathrm{Im}(2\tau))| \\ &\qquad + 2(196884e^{2\pi \mathrm{Im}(\tau)} + e^{2\pi \mathrm{Im}(\tau)})|f(\mathrm{Im}(\tau))| + |f(\mathrm{Im}(\tau))|^2 \\ &\quad \le 10^{-7} + |f(6)| + 10^6|f(3)| + |f(3)|^2 \\ &\quad \le 0.4 \end{aligned}$$

We now let

$$\gamma_0 = \tilde{j}^2 - 744\tilde{j} + 19688 - 744^2 + 744.$$

This will be our starting point for Newton iteration to find β_0. We now bound the terms appearing in Kantorovich's theorem. Firstly,

$$\begin{aligned}\Phi_2(\tilde{j}, z) = z^3 &+ (-\tilde{j}^2 + 1488\tilde{j} - 162000)z^2 \\ &+ (1488\tilde{j}^2 + 40773375\tilde{j} + 8748000000)z \\ &+ \tilde{j}^3 - 162000\tilde{j}^2 + 8748000000\tilde{j} - 157464000000000.\end{aligned}$$

Now it is clear, as $|j - \tilde{j}| \le 2^{-300}|j|$, that $|\gamma_0 - j(2\tau)| \le 0.4 + \frac{|j|^2}{2^{90}}$, and so as $0.98|j|^2 \le |j(2\tau)| \le 1.02|j|^2$, we have $0.979|j|^2 \le |\gamma_0| \le 1.021|j|^2$. we will take the r of Kantorovich's theorem to be $0.009|j|^2$, and give bounds for the disc $|\gamma - \gamma_0| \le 0.009|j|^2$. For γ in this disc, $0.97|j|^2 \le |\gamma| \le 1.03|j|^2$ and in addition $|\tilde{j}| \le 1.01|j|$, so for an upper bound on the first derivative, we have

$$\begin{aligned}|\Phi_2'(\tilde{j}, \gamma)| &\le 3|\gamma|^2 + 2(|\tilde{j}|^2 + 1488|\tilde{j}| + 162000)|\gamma_0| \\ &\quad + (1488|\tilde{j}|^2 + 40773375|\tilde{j}| + 8748000000) \\ &\le 3.2|j|^4 + 2.2|j|^4 + 10^{-7}|j|^4 \\ &\le 5.5|j|^4,\end{aligned}$$

and for a lower bound, we have

$$\begin{aligned}|\Phi_2'(\tilde{j}, \gamma)| &\ge 3|\gamma|^2 - 2(|\tilde{j}|^2 + 1488|\tilde{j}| + 162000)|\gamma_0| \\ &\quad - (1488|\tilde{j}|^2 + 40773375|\tilde{j}| + 8748000000) \\ &\ge 2.82|j|^4 - 2.1|j|^4 - 10^{-7}|j|^4 \\ &\ge 0.71|j|^4.\end{aligned}$$

For our upper bound on the function at γ_0, as $|\beta_1|, |\beta_2| \le 2|j|^{1/2}$, and as $|\beta_0 - j(2\tau)| \le 2^{-298}|j|^2$ we have $|\gamma_0 - \beta_0| \le 0.4 + 2^{-298}|j|^2$, so

$$\begin{aligned}|\Phi_2(\tilde{j}, \gamma_0)| &= |\gamma_0 - \beta_0||\gamma_0 - \beta_1||\gamma_0 - \beta_2| \\ &\le (0.4 + 2^{-289}|j|^2)(1.03|j|^2 + 2|j|^{1/2})^2 \\ &\le 2^{-54}|j|^6.\end{aligned}$$

For the second derivative, we have for an upper bound

$$\begin{aligned}|\Phi_2''(\tilde{j}, \gamma)| &\le 6|\gamma| + 2|\tilde{j}|^2 + 1488|\tilde{j}| + 162000 \\ &\le 8.3|j|^2,\end{aligned}$$

and for a lower bound,

$$\begin{aligned}|\Phi_2''(\tilde{j}, \gamma_0)| &\ge 6|\gamma| - 2|\tilde{j}|^2 - 1488|\tilde{j}| - 162000 \\ &\ge 3.8|j|^2.\end{aligned}$$

Now, by Theorem 3, as

$$\frac{|\Phi_2(\tilde{j}, \gamma_0)||\Phi_2''(\tilde{j}, \gamma)|}{|\Phi_2'(\tilde{j}, \gamma)|^2} \le \frac{2^{-54}|j|^6 \cdot 8.3|j|^2}{(0.71|j|^4)^2} \le 2^{-49} < \frac{1}{2},$$

and

$$2\eta = 2\frac{|\Phi_2(\tilde{j},\gamma_0)|}{|\Phi_2'(\tilde{j},\gamma_0)|} \le \frac{2^{-54}|j|^6}{0.71|j|^4} \le 0.001|j|^2,$$
$$r = 0.009|j|^2,$$

Newton's method will converge to the root β_0, with a rate of convergence

$$|\gamma_k - \beta_0| \le \frac{1}{2^k}2^{-48\cdot 2^k}\frac{|\Phi_2'(\tilde{j},\gamma_0)|}{|\Phi_2''(\tilde{j},\gamma_0)|} \le \frac{1}{2^k}2^{-48\cdot 2^k}\frac{5.5|j|^4}{3.8|j|^2} \le 2^{-48\cdot 2^k}|j|^2$$

for $k \ge 1$. In particular, when $k \ge [2\log P + 2\log\log|j|]$, $|\gamma_k - \beta_0| \le 2^{-P}$. Now as, by Proposition 1, β_0 is an approximation to $j(2\tau)$ of relative precision 2^{-P+2}, it may be easily shown that after $[2\log P + 2\log\log|j|]$ steps, γ_k will be an approximation to $j(2\tau)$ of relative precision 2^{-P+3}. □

Lemma 2. *Suppose that* $|j(\tau)| \ge 2^{P+12}$, *and* $\tilde{j}$ *is an approximation to* $j(\tau)$ *of relative precision at least* 2^{-P}. *Then*

$$-\frac{\log\tilde{j}}{2\pi}$$

is an approximation to τ *of absolute precision* 2^{-P}.

Proof. Let $j(\tau) + \delta = \tilde{j}$. As

$$|j(\tau) - e^{-2\pi\tau}| \le 2079,$$

we have

$$\log(\tilde{j}) = -2\pi\tau + \log\left(1 + \frac{\delta + 2079\theta}{j(\tau)}\right),$$

where $|\theta| \le 1$. So as $|\log(1+z)| \le 1.1|z|$ when $|z| \le 0.05$,

$$|\log\tilde{j} + 2\pi\tau| \le 1.1\cdot 2^{-P} + 0.6\cdot 2^{-P} \le 1.7\cdot 2^{-P}.$$

So that

$$\left|\tau - \left(-\frac{\log\tilde{j}}{2\pi}\right)\right| \le \frac{1.7}{2\pi}2^{-P} \le 2^{-P}.$$

□

Now the algorithm to invert j when $\mathrm{Im}(\tau) \ge 3$ proceeds as follows – if $|j| \le 2^{P+12}$, first iteratively compute approximations to $j(2^k\tau)$ by Newton's method applied to $\Phi_2(\tilde{j}, z)$, up to $k = \left[2\log\left(\frac{P+12}{\mathrm{Im}(\tau)}\right)\right] + 1$. Then calculate the logarithm of $j(2^k\tau)$ to relative precision 2^{-P-2} (note that $|\tau| \ge 1$), and divide by -2π, where we have calculated 2π to relative precision 2^{-P-2}. This process entails a loss of precision of at most $5\left[2\log\left(\frac{P+12}{\mathrm{Im}(\tau)}\right)\right] + 2$, which is bounded by $11\log P$ when $P \ge 400$, and the precision at all applications of Newton's method is at least 2^{-300}, so our assumptions on the precision of our approximations in the propositions of this section are satisfied at each application. As the computational complexity of Newton's method is $O(M(P)\log P)$, and of the complex logarithm and computing π are $O(M(P)(\log P)^2)$, if $|j| \le 2^{P+12}$, the algorithm has time complexity

$$O\left(M(P)\left[2\log\left(\frac{P+12}{\mathrm{Im}(\tau)}\right) + 1\right][2\log P + 2\log\log|j|]\right) = O\left(M(P)(\log P)^2\right),$$

and if $|j| \ge 2^{P+12}$, time complexity

$$O\left(M(P)(\log P)^2\right),$$

where the implicit constants are not too large and could be made effective.

3.2 Near 1728 and 0

When τ is close to either i or $\frac{\pm 1+i\sqrt{3}}{2}$, we will make use of the modular polynomial $\Phi_2(X,Y)$ to obtain an approximation to one of $j(2\tau)$, $j\left(\frac{\tau}{2}\right)$, or $j\left(\frac{\tau+1}{2}\right)$, for which the $\mathrm{SL}_2(\mathbb{Z})$–equivalent elements of $\mathcal{F}$ to either 2τ, $\frac{\tau}{2}$, or $\frac{\tau+1}{2}$ will lie in the aforementioned compact set, to which we may apply Newton's method. We carry out the analysis only for i and $\frac{1+i\sqrt{3}}{2}$, as when $\left|\tau - \frac{-1+i\sqrt{3}}{2}\right| \leq 2^{-31}$, the $\mathrm{SL}_2(\mathbb{Z})$–equivalent $\tau+1$ satisfies $\left|(\tau+1) - \frac{1+i\sqrt{3}}{2}\right| \leq 2^{-31}$.

Lemma 3. *If* $|\delta| \leq 2^{-30}$ *then*

$$\left| j(i+\delta) - 1728 - \frac{j^{(2)}(i)}{2}\delta^2 \right| \leq 0.07|\delta|^2,$$

and

$$\left| j\left(\frac{1+i\sqrt{3}}{2} + \delta\right) - \frac{j^{(3)}\left(\frac{1+i\sqrt{3}}{2}\right)}{3!}\delta^3 \right| \leq 0.07|\delta|^3.$$

Proof. We first bound the coefficients of the Taylor series of j at $z = i$ and $z = \frac{1+i\sqrt{3}}{2}$. Firstly, by Theorem 1 of [4], the coefficient of $e^{2\pi i n\tau}$ in the q–expansion of j is bounded by $4e^{4\pi\sqrt{n}}$, so the corresponding coefficient of the k'th derivative is bounded by $(2\pi n)^k e^{4\pi\sqrt{n}}$. Further, for $n \geq 1$, $e^{4\pi\sqrt{n} - \sqrt{3}\pi n} \leq 100e^{-\pi n}$, so we have the following bound of the derivatives at these two points,

$$|j^{(k)}(i)|, \left| j^{(k)}\left(\frac{1+i\sqrt{3}}{2}\right)\right| \leq (2\pi)^k e^{2\pi} + 744 + 400\sum_{n=1}^{\infty}(2\pi n)^k e^{-\pi n}.$$

We now bound the sum in this expression. Firstly, we have the identity

$$\sum_{n=1}^{\infty} e^{-\pi n x} = \frac{1}{e^{\pi x} - 1},$$

and so consider the derivatives of this function, expressing the derivative as sums of the form

$$\sum \alpha_i \frac{e^{a\pi x}}{(e^{\pi x} - 1)^b},$$

where each term occurring in the expression for the k'th derivative is derived from taking the derivative of either the numerator or denominator of a term occurring in the $k-1$'th derivative, i.e. there is no collection of terms with (a,b) equal. It is clear that there are at most 2^k terms, and, passing from one derivative to the next, $|\alpha_i|$ may increase by at most $\pi \max\{a,b\}$, and that $a \leq b \leq k+1$, and $a \leq k$. So $|\alpha_i| \leq \pi^k (k+1)!$, and a bound for the whole expression evaluated at 1 is therefore

$$2^k \pi^k (k+1)! \frac{e^{k\pi}}{(e^{\pi} - 1)^k} \leq 14^k k!.$$

Now we have the bounds

$$\begin{aligned} |j^{(k)}(i)|, \left| j^{(k)}\left(\frac{1+i\sqrt{3}}{2}\right)\right| &\leq (2\pi)^k e^{2\pi} + 744 + 400 \cdot 14^k k! \\ &\leq 1700 \cdot 14^k k!. \end{aligned}$$

At $z = i$ and $z = \frac{1+i\sqrt{3}}{2}$, the Taylor series expansions for $j(z)$ are

$$j(z) = 1728 + \sum_{n=2}^{\infty} c_n (z-i)^n,$$
$$j(z) = \sum_{n=3}^{\infty} c'_n \left(z - \frac{1+i\sqrt{3}}{2} \right)^n .$$

where c_n and c'_n are bounded in absolute value by $1400 \cdot 14^k$. We now bound the tails of the sums of the Taylor series,

$$\sum_{n=3}^{\infty} |c_n||\delta|^n \leq |\delta|^3 \sum_{n=0}^{\infty} 1700 \cdot 14^{n+3} |\delta|^n,$$

and

$$\sum_{n=4}^{\infty} |c'_n||\delta|^n \leq |\delta|^4 \sum_{n=0}^{\infty} 1700 \cdot 14^{n+4} |\delta|^n.$$

Now as $|\delta| \leq 2^{-30}$,

$$\sum_{n=0}^{\infty} 1700 \cdot 14^{n+4} |\delta|^n = \frac{1700 \cdot 14^4}{1 - 14|\delta|} \leq 7 \cdot 10^7,$$

so that

$$\left| j(i+\delta) - 1728 - \frac{j^{(2)}(i)}{2} \delta^2 \right| \leq 2^{-30} \cdot 7 \cdot 10^7 |\delta|^2 \leq 0.07 |\delta|^2,$$
$$\left| j\left(\frac{1+i\sqrt{3}}{2} + \delta \right) - \frac{j^{(3)}\left(\frac{1+i\sqrt{3}}{2}\right)}{3!} \delta^3 \right| \leq 2^{-30} \cdot 7 \cdot 10^7 |\delta|^3 \leq 0.07 |\delta|^3.$$

□

Furthermore, we note that $49700 \geq |j^{(2)}(i)| \geq 49600$, and $275000 \geq \left| j^{(3)}\left(\frac{1+i\sqrt{3}}{2}\right) \right| \geq 274000$, so if $|\tau - i| \leq 2^{-30}$, by the previous lemma, we have

$$2.4 \cdot 10^4 |\delta|^2 \leq |j - 1728| \leq 2.5 \cdot 10^4 |\delta|,$$

and if $\left| \tau - \frac{1+i\sqrt{3}}{2} \right| \leq 2^{-30}$,

$$4.5 \cdot 10^4 |\delta|^3 \leq |j| \leq 4.6 \cdot 10^4 |\delta|^3,$$

from which we deduce the following,

Lemma 4. *If $|\tau - i| \leq 2^{-30}$ and $P \geq 300$, then:*

1. *If $|j(\tau) - 1728| \leq 2^{-P}$, then $|\tau - i| \leq 2^{-P/2-7}$.*
2. *If $|j(\tau) - 1728| \geq 2^{-P}$, then $|\tau - i| \geq 2^{-P/2-8}$.*

If $\left| \tau - \frac{1+i\sqrt{3}}{2} \right| \leq 2^{-30}$ and $P \geq 300$, then:

1. *If $|j| \leq 2^{-P}$, then $\left| \tau - \frac{1+i\sqrt{3}}{2} \right| \leq 2^{-P/3-5}$.*

2. If $|j| \geq 2^{-P}$*, then* $\left|\tau - \frac{1+i\sqrt{3}}{2}\right| \geq 2^{-P/3-6}$.

Lemma 5. *Suppose that* $|\delta| \leq 2^{-28}$*. Then*

$$|j\left(2i+\delta\right) - j\left(2i\right) - j'\left(2i\right)\delta| \leq 0.2|\delta|,$$

and

$$\left|j\left(i\sqrt{3}+\delta\right) - j\left(i\sqrt{3}\right) - j'\left(i\sqrt{3}\right)\delta\right| \leq 0.2|\delta|.$$

Proof. We proceed similarly to the previous lemma – as $17e^{-2\pi n} \geq e^{4\pi n - 2\sqrt{3}\pi n}$,

$$|j^{(k)}(2i)|, |j^{(k)}(\sqrt{3}i)| \leq (2\pi)^k e^{4\pi} + 744 + 68\sum_{n=1}^{\infty}(2\pi n)^k e^{-2\pi n},$$

and a similar analysis to the previous lemma bounds the sum in this inequality by

$$(2\pi)^k e^{4\pi} + 744 + 68 \cdot 13^k k! \leq 300000 \cdot 13^k k!. \tag{3}$$

So when $|\delta| \leq 2^{-28}$,

$$\begin{aligned}
|j\left(2i+\delta\right) - j(2i) - j'\left(2i\right)\delta| &\leq \sum_{n=2}^{\infty} 300000 \cdot 13^n \delta^n \\
&\leq \sum_{n=2}^{\infty} 300000 \cdot 13^n \delta^n \\
&\leq 2^{-28} \cdot 300000 \cdot 13^2 |\delta| \sum_{n=0}^{\infty} 13^n 2^{-28n} \\
&\leq 0.2|\delta|.
\end{aligned}$$

and as our bound for the terms of the Taylor series applies to both expansions, we similarly have

$$\left|j\left(i\sqrt{3}+\delta\right) - j\left(i\sqrt{3}\right) - j'\left(i\sqrt{3}\right)\delta\right| \leq 0.2|\delta|.$$

□

We now give two lemmas on the separation and closeness of the three preimages of roots of the modular polynomial $\Phi_2(j(\tau), z)$, for τ near i and $\frac{1+i\sqrt{3}}{2}$.

Lemma 6. *Suppose that* $\tau \in \mathcal{F}$*. Then, if* $\tau = i + \delta$*,*

$$|2\tau - 2i| \leq 2|\delta|,$$
$$\left|-\frac{2}{\tau} - 2i\right| \leq 2|\delta|,$$

and if $\tau = \frac{1+i\sqrt{3}}{2} + \delta$*,*

$$|(2\tau - 1) - \sqrt{3}i| \leq 2|\delta|,$$
$$\left|\left(1 - \frac{2}{\tau}\right) - \sqrt{3}i\right| \leq 2|\delta|,$$
$$\left|\left(-1 - \frac{2}{\tau - 1}\right) - \sqrt{3}i\right| \leq 2|\delta|.$$

Proof. First note that as $\tau \in \mathcal{F}$, $|\tau| \geq 1$. Let $2\tau = 2i + \delta_1$, and $-\frac{2}{\tau} = 2i + \delta_2$. For δ_1, $|2\tau - 2i| = 2|\delta|$, and for δ_2,

$$\begin{aligned} |\delta_2| = \left|-\frac{2}{\tau} - 2i\right| &= \left|\frac{-2-2i\tau}{\tau}\right| \\ &= \frac{|2i\delta|}{|\tau|} \\ &\leq 2|\delta| \end{aligned}$$

If $\tau = \frac{1+i\sqrt{3}}{2} + \delta$, let $2\tau - 1 = \sqrt{3}i + \delta_1$, $\left(1 - \frac{2}{\tau}\right) = \sqrt{3}i + \delta_2$, and $\left(1 - \frac{2}{\tau - 1}\right) = \sqrt{3}i + \delta_3$. For δ_1,

$$|\delta_1| = \left|2\tau - 1 - \sqrt{3}i\right| \leq 2|\delta|,$$

for δ_2,

$$\begin{aligned} |\delta_2| = \left|\left(1 - \frac{2}{\tau}\right) - \sqrt{3}i\right| &= \left|\frac{\tau - 2 - \sqrt{3}i\tau}{\tau}\right| \\ &= \left|\frac{(1-\sqrt{3}i)\delta}{\tau}\right| \\ &\leq 2|\delta|, \end{aligned}$$

and for δ_3,

$$\begin{aligned} |\delta_3| = \left|\left(-1 - \frac{2}{\tau - 1}\right) - \sqrt{3}i\right| &= \left|\frac{-\tau + 1 - 2 - \sqrt{3}i\tau + \sqrt{3}i}{\tau}\right| \\ &= \frac{|(1+\sqrt{3}i)\delta|}{|\tau - 1|} \\ &\leq 2|\delta|. \end{aligned}$$

□

Lemma 7. *If $\tau = i + \delta$, $\tau \in \mathcal{F}$, with $|\delta| \leq 2^{-30}$, then the distance between 2τ and $-\frac{2}{\tau}$ is at least $3.99|\delta|$, and the distance between either of these and the point in $\mathcal{F}$ which is $\mathrm{SL}_2(\mathbb{Z})$–equivalent to $\frac{\tau+1}{2}$ is at least 0.99 in magnitude. If $\tau = \frac{1+i\sqrt{3}}{2} + \delta$, $\tau \in \mathcal{F}$, then the distance between any pair of the three points lying in $\mathcal{F}$ which are $\mathrm{SL}_2(\mathbb{Z})$–equivalent to 2τ, $\frac{\tau}{2}$, and $\frac{\tau+1}{2}$ is at least $3.46|\delta|$.*

Proof. Firstly, for the distance between τ and $-\frac{2}{\tau}$, we have

$$\begin{aligned} 2\tau + \frac{2}{\tau} &= \frac{2(\tau^2+1)}{\tau} \\ &= \frac{4i\delta + 2\delta^2}{\tau}. \end{aligned}$$

As $|\delta| \leq 2^{-30}$, $|\delta|^2 \leq 2^{-30}|\delta|$, and $|i + \delta| \leq 1 + 2^{-30}$, so

$$\left|2\tau - \left(-\frac{2}{\tau}\right)\right| \geq 3.99|\delta|.$$

Now for $-\frac{2}{\tau+1}+1$, we have

$$\begin{aligned} -\frac{2}{\tau+1}+1-i &= \frac{-2+\tau+1-i\tau-i}{\tau+1} \\ &= \frac{\delta-i\delta}{1+i+\delta} \end{aligned}$$

so that

$$\left|\left(-\frac{2}{\tau+1}+1\right)-i\right| \leq 2|\delta| \tag{4}$$

and so either $-\frac{2}{\tau+1}+1$ or $-\left(-\frac{2}{\tau+1}+1\right)^{-1}$ is $\mathrm{SL}_2(\mathbb{Z})$–equivalent to $\frac{1+\tau}{2}$ and in $\mathcal{F}$, and its distance from i is bounded by $\left|1+\frac{2|\delta|}{1-2|\delta|}\right| \leq 2^{-28}$. By Lemma 6, the distance of either 2τ or $-\frac{2}{\tau}$ from $2i$ is at most $2|\delta| \leq 2\cdot 2^{-30}$, so both of these are separated from either $-\frac{2}{\tau+1}+1$ or $-\left(-\frac{2}{\tau+1}+1\right)^{-1}$ by a distance of at least 0.99.

Now for $\tau=\frac{1+i\sqrt{3}}{2}+\delta$, the $\mathrm{SL}_2(\mathbb{Z})$–equivalent points to 2τ, $\frac{\tau}{2}$, and $\frac{\tau+1}{2}$ are $2\tau-1$, $1-\frac{2}{\tau}$, and $-1-\frac{2}{\tau-1}$ respectively. For their differences, we have,

$$\begin{aligned} 2\tau-1-\left(1-\frac{2}{\tau}\right) &= \frac{2(\tau^2-\tau+1)}{\tau} \\ &= \frac{i2\sqrt{3}\delta+2\delta^2}{\tau}. \end{aligned}$$

So as $|\tau| \leq 1+2^{-30}$,

$$\left|2\tau-1-\left(1-\frac{2}{\tau}\right)\right| \geq 3.46|\delta|$$

For the first and third points,

$$\begin{aligned} 2\tau-1-\left(-1-\frac{2}{\tau-1}\right) &= \frac{2(\tau^2-\tau+1)}{\tau-1} \\ &= \frac{i2\sqrt{3}\delta+2\delta^2}{\tau-1}. \end{aligned}$$

As $|\tau-1| \leq 1+2^{-30}$, we have the lower bound

$$\left|2\tau-1-\left(-1-\frac{2}{\tau-1}\right)\right| \geq 3.46|\delta|.$$

For the second and third points,

$$\begin{aligned} \left(1-\frac{2}{\tau}\right)-\left(-1-\frac{2}{\tau-1}\right) &= \frac{2(\tau^2-\tau+1)}{\tau(\tau-1)} \\ &= \frac{i2\sqrt{3}\delta+2\delta^2}{\tau(\tau-1)}. \end{aligned}$$

So we obtain the lower bound

$$\left|\left(1-\frac{2}{\tau}\right)-\left(-1-\frac{2}{\tau-1}\right)\right| \geq 3.46|\delta|.$$

□

Lemma 8. *If* $\tau = i + \delta$*, with* $|\delta| \le 2^{-30}$*, then*

$$7.18 \cdot 10^6 |\delta| \le \left| j(2\tau) - j\left(\frac{\tau}{2}\right) \right| \le 7.25 \cdot 10^6 |\delta|,$$

and

$$2.4 \cdot 10^5 \le \left| j(2\tau) - j\left(\frac{\tau+1}{2}\right) \right|, \left| j\left(\frac{\tau}{2}\right) - j\left(\frac{\tau+1}{2}\right) \right| \le 3.1 \cdot 10^5,$$

and if $\tau = \frac{1+i\sqrt{3}}{2} + \delta$*, with* $|\delta| \le 2^{-30}$*, then for* $\tau_1, \tau_2 \in \{2\tau, \frac{\tau}{2}, \frac{\tau+1}{2}\}$*,* $\tau_1 \ne \tau_2$*,*

$$1.15 \cdot 10^6 |\delta| \le |j(\tau_1) - j(\tau_2)| \le 1.34 \cdot 10^6 |\delta|.$$

Proof. We first use the Taylor series expansion of $j(z)$ at $2i$. By Lemma 6, we may apply Lemma 5, and using also Lemma 7, we obtain the lower bound

$$\begin{aligned}
\left| j(2\tau) - j\left(-\frac{2}{\tau}\right) \right| &\ge |j(2i) + j'(2i)\delta_1 - j(2i) - j'(2i)\delta_2| - 0.4|\delta| \\
&= |j'(2i)|\,|\delta_1 - \delta_2| - 0.4|\delta| \\
&= |j'(2i)| \left| 2\tau + \frac{2}{\tau} \right| - 0.4|\delta| \\
&\ge 1.8 \cdot 10^6 \cdot 3.99|\delta| - 0.4|\delta| \\
&\ge 7.18 \cdot 10^6 |\delta|.
\end{aligned}$$

Similarly we have an upper bound of

$$\left| j(2\tau) - j\left(-\frac{2}{\tau}\right) \right| \le 1.81 \cdot 10^6 \cdot 4|\delta| + 194|\delta| \le 7.25 \cdot 10^6 |\delta|.$$

By Lemma 7, the $\mathrm{SL}_2(\mathbb{Z})$–equivalent point in $\mathcal{F}$ to $\frac{\tau+1}{2}$ is at most $2|\delta| \le 2^{-29}$ from i, so

$$\left| j\left(\frac{\tau+1}{2}\right) \right| \le e^{2.01\pi} + 2079 \le e^{7.9},$$

and by Lemma 6, $2\tau, -\frac{\tau}{2}$ are at most 2^{-29} from $2i$,

$$|j(2\tau)|, \left| j\left(\frac{\tau}{2}\right) \right| \ge e^{3.99\pi} - 2079 \ge e^{12.5},$$

so that

$$3.1 \cdot 10^5 \ge \left| j(2\tau) - j\left(\frac{\tau+1}{2}\right) \right|, \left| j\left(-\frac{\tau}{2}\right) - j\left(\frac{\tau+1}{2}\right) \right| \ge 2.4 \cdot 10^5.$$

Now using the Taylor series of $j(z)$ at $i\sqrt{3}$, by Lemma 6, we may apply Lemma 5, and using also Lemma 7, with τ_1, τ_2 any distinct pair of 2τ ($\mathrm{SL}_2(\mathbb{Z})$–equivalent to $2\tau - 1$), $\frac{\tau}{2}$, ($\mathrm{SL}_2(\mathbb{Z})$–equivalent to $1 - \frac{2}{\tau}$), and $\frac{\tau+1}{2}$ ($\mathrm{SL}_2(\mathbb{Z})$–equivalent to $-1 - \frac{1}{\tau - 1}$), with $\tau_j = i\sqrt{3} + \delta_j$,

$$\begin{aligned}
|j(\tau_1) - j(\tau_2)| &\ge \left| j(\sqrt{3}i) + j'(\sqrt{3}i)\delta_1 - (j(\sqrt{3}i) + j'(\sqrt{3}i)\delta_2) \right| - 0.4|\delta| \\
&= |j'(\sqrt{3}i)||\tau_1 - \tau_2| - 0.4|\delta| \\
&\ge 334500 \cdot 3.46|\delta| - 0.4|\delta| \\
&\ge 1.15 \cdot 10^6 |\delta|,
\end{aligned}$$

and similarly for an upper bound we have

$$\begin{aligned}|j(\tau_1) - j(\tau_2)| &\le \left| j(\sqrt{3}i) + j'(\sqrt{3}i)\delta_1 - (j(\sqrt{3}i) + j'(\sqrt{3}i)\delta_2) \right| + 0.4|\delta| \\ &= |j'(\sqrt{3}i)||\tau_1 - \tau_2| + 0.4|\delta| \\ &\le 334600 \cdot 4|\delta| + 0.4|\delta| \\ &\le 1.34 \cdot 10^6 |\delta|.\end{aligned}$$

□

3.2.1 j near 1728

We now bound the discrepancy between the roots of $\Phi_2(j, z)$ and $\Phi_2(\tilde{j}, z)$ when $|\tau - i| \le 2^{-20}$ and $|\tilde{j} - 1728| \ge 2^{-P/3}$.

Lemma 9. *Suppose that $\tilde{j}$ is an approximation to $j(\tau)$ of relative precision 2^{-P}, with $P \ge 300$, $|\tau - i| \le 2^{-30}$, and $\left|\tilde{j} - 1728\right| \ge 2^{-P/3}$. Then the relative precision of any root of $\Phi_2(\tilde{j}, z)$ to the closest root of $\Phi_2(j, z)$ is at least $2^{-P/3+10}$*

Proof. Firstly, as in the proof of Lemma 1, with $f(z) = \Phi_2(j, z)$ and $g(z) = \Phi_2(\tilde{j}, z)$, and $\tilde{j} = j + \delta$, we will bound the size of the coefficients of $f - g$. We note that $|j| \ge 1700$. For z^2, we have

$$|2j\delta + \delta^2 + 1448\delta| \le 3|\delta||j|,$$

for z, we have

$$|2976\delta j + 1488\delta^2 + 40773375\delta| \le 30000|\delta||j|,$$

and for the constant term

$$\left|3\delta j^2 + 3\delta^2 j + \delta^3 - 16200\delta j - 16200\delta^2 + 8748000000\delta\right| \le 3100|\delta||j|^2.$$

Now evaluating g at $j(2\tau)$, we have, noting that $|j(2\tau)| \le 0.1|j|^2$,

$$\begin{aligned}|g(j(2\tau))| &= |f(j(2\tau)) - g(j(2\tau))| \\ &\le 3|\delta||j|(0.1|j|^2)^2 + 30000|\delta||j|(0.1|j|^2) + 3100|\delta||j|^2 \\ &\le 0.04|j|^5\end{aligned}$$

Letting $\beta_0, \beta_1, \beta_2$ be the roots of g, where β_0 is the closest root of $g(z)$ to $j(2\tau)$,

$$\begin{aligned}|j(2\tau) - \beta_0||j(2\tau) - \beta_1||j(2\tau) - \beta_2| &\le 0.04|\delta||j|^5 \\ |j(2\tau) - \beta_0| &\le 10^5|\delta|^{1/3} \\ &\le 1.2 \cdot 10^6 \cdot 2^{-P/3},\end{aligned}$$

and similarly for the nearest root of $g(z)$ to each of the roots of $f(z)$. As $|\tilde{j} - 1728| \ge 2^{-P/3}$, $|j - 1728| \ge 2^{-P/3} - 1800 \cdot 2^{-P} \ge 2^{-P/3-1}$, and by Lemma 4, $|\tau - i| \ge 2^{-P/6-9}$, so that by Lemma 8,

$$\left| j(2\tau) - j\left(\frac{\tau}{2}\right) \right| \ge 7.18 \cdot 10^7 \cdot 2^{-P/6-9}.$$

Now letting β_i and β_j be the nearest roots to $j(2\tau)$ and $j\left(\frac{\tau}{2}\right)$ respectively,

$$\begin{aligned}
|\beta_i - \beta_j| &= \left|(\beta_i - j(2\tau)) - j(2\tau) - \left(\left(\beta_j - j\left(\frac{\tau}{2}\right)\right) - j\left(\frac{\tau}{2}\right)\right)\right| \\
&\geq \left|j(2\tau) - j\left(\frac{\tau}{2}\right)\right| - |j(2\tau) - \beta_i| - \left|j\left(\frac{\tau}{2}\right) - \beta_j\right| \\
&\geq 7.18 \cdot 10^7 \cdot 2^{-P/6-9} - 1.2 \cdot 10^6 \cdot 2^{-P/3} \\
&\geq 7.18 \cdot 10^7 \cdot 2^{-P/6-9} - 0.01 \cdot 2^{-P/6} \\
&\geq 1.4 \cdot 10^5 \cdot 2^{-P/6}.
\end{aligned}$$

In particular, β_i and β_j are distinct, and given that, by Lemma 8, $\left|j(2\tau) - j\left(\frac{\tau+1}{2}\right)\right| \geq 2.4 \cdot 10^5 \geq 7.18 \cdot 10^7 \cdot 2^{-P/6-9}$, one may similarly deduce a separation of $2 \cdot 10^5$ between the closest root to $j\left(\frac{\tau+1}{2}\right)$ and either of the other two. So each root j_i of $\Phi_2(j, z)$ has a unique associated closest root β_i of $\Phi_2(\tilde{j}, z)$, which satisfies

$$|j_i - \beta_i| \leq 1.2 \cdot 10^6 \cdot 2^{-P/3}.$$

The relative precision of β_i to its closest root is then bounded by

$$\frac{1.2 \cdot 10^6 \cdot 2^{-P/3}}{|j(2\tau)|}, \frac{1.2 \cdot 10^6 \cdot 2^{-P/3}}{\left|j\left(\frac{\tau}{2}\right)\right|} \leq 2^{-P/3+3}$$

if β_i is the root close to $j(2\tau)$ or $j\left(\frac{\tau}{2}\right)$, and

$$\frac{1.2 \cdot 10^6 \cdot 2^{-P/3}}{|j(\frac{\tau+1}{2})|} \leq 2^{-P/3+10}$$

for the root close to $j\left(\frac{\tau+1}{2}\right)$. □

Finally we show that Newton's method may be used to obtain an approximation to $j(2\tau)$.

Proposition 3. *Let $j = j(\tau)$, where $|\tau - i| \leq 2^{-30}$, and $\tilde{j}$ be an approximation to j of relative precision 2^{-P}, $P \geq 300$, such that $|\tilde{j} - 1728| \geq 2^{-P/3}$. Let*

$$\tau_0 = i + \sqrt{\frac{\tilde{j} - 1728}{j^{(2)}(i)}},$$

where the sign of the square root is chosen arbitrarily. Then Newton's method applied to $\Phi_2(\tilde{j}, z)$, with starting point $j(2\tau_0)$ will converge to either the root of $\Phi_2(\tilde{j}, z)$ closest to $j(2\tau)$, or the root of $\Phi_2(\tilde{j}, z)$ closest to $j\left(\frac{\tau}{2}\right)$, and after $\lceil 2 \log P \rceil$ steps will produce an approximation either $j(2\tau)$ or $j\left(\frac{\tau}{2}\right)$ of relative precision $2^{-P/3+11}$.

Proof. Firstly, we bound the difference between τ and τ_0. We let $\tau = i + \delta$. By Lemma 3,

$$\left|j(\tau) - 1728 - \frac{j^{(2)}(i)}{2}\delta^2\right| \leq 0.07|\delta|^2,$$

so that

$$\left|\sqrt{\frac{2(\tilde{j} - 1728)}{j^{(2)}(i)}} - \delta\right|\left|\sqrt{\frac{2(\tilde{j} - 1728)}{j^{(2)}(i)}} + \delta\right| \leq \frac{388}{j^{(2)}(i)}|\delta|^2 + \frac{2^{-P+1}|j|}{j^{(2)}(i)} \leq 3 \cdot 10^{-6}|\delta|^2.$$

Now we will show that, taking some branch of the square root above, we will obtain a good starting point for Newton's method. Let $\epsilon = \sqrt{\frac{2(\tilde{j}-1728)}{j^{(2)}(i)}}$, where the branch of the square root is arbitrary. Firstly, one of $|\epsilon - \delta|$ or $|\epsilon + \delta|$ is $\geq |\delta|$, and so the other is $\leq 3 \cdot 10^{-6}|\delta|$. In the first case, $|\tau_0 - \tau| \leq 3 \cdot 10^{-6}|\delta|$, and in the second case,

$$\left|\tau_0 - \left(-\frac{1}{\tau}\right)\right| = \left|\frac{-1 + i\delta + i\epsilon + 1 + \delta\epsilon}{\tau}\right| \leq |\delta + \epsilon| + |\delta\epsilon| \leq 3.1 \cdot 10^{-6}|\delta|.$$

Now as $|\delta| \leq 2^{-30}$, $j'(z)$ is bounded in absolute value by $1.81 \cdot 10^6$ between $2\tau_0$ and either of 2τ or $-\frac{2}{\tau}$, and the distance from $2\tau_0$ to the closest of these is bounded by $6.2 \cdot 10^{-6}|\delta|$, so either

$$|j(2\tau_0) - j(2\tau)| \leq 12|\delta|$$

or

$$\left|j(2\tau_0) - j\left(-\frac{2}{\tau}\right)\right| \leq 12|\delta|.$$

Now we bound the terms occurring in Kantorovich's criterion. Let $\beta_0, \beta_1, \beta_2$ be the roots of $\Phi_2(\tilde{j}, z)$, with β_0 the closest root to $j(2\tau_0)$, and β_1 the other root near $j(2i)$. Firstly, as $|j| \geq 2^{-P/2}$, by Lemma 8, the above, and Lemma 9, we have, for $2\tau_0$,

$$\begin{aligned}
|j(2\tau_0) - \beta_0| &\leq 12|\delta| + 2^{-P/3+10} \leq 12.1|\delta| \\
|j(2\tau_0) - \beta_1| &\leq 7.25 \cdot 10^6|\delta| + 2^{-P/3+10} \leq 7.26 \cdot 10^6|\delta| \\
|j(2\tau_0) - \beta_2| &\leq 3.1 \cdot 10^5 + 2^{-P/3+10} \leq 3.11 \cdot 10^5 \\
|j(2\tau_0) - \beta_1| &\geq 7.18 \cdot 10^6|\delta| - 2^{-P/3+10} \geq 7.17 \cdot 10^6|\delta| \\
|j(2\tau_0) - \beta_2| &\geq 2.4 \cdot 10^5 - 2^{-P/3+10} \geq 2.39 \cdot 10^5,
\end{aligned}$$

and similarly, for any τ' satisfying $|\tau' - 2\tau| \leq 35|\delta|$ (which we take to ensure the condition on r is satisfied), we have, as $|j'(z)| \leq 1.81 \cdot 10^6$ between $2\tau_0$ and τ', and as $|\delta| \leq 2^{-30}$,

$$\begin{aligned}
|j(\tau') - \beta_0| &\leq 6.4 \cdot 10^7|\delta| + 12|\delta| \leq 0.06, \\
|j(\tau') - \beta_1| &\leq 6.4 \cdot 10^7|\delta| + 7.26 \cdot 10^6|\delta| \leq 0.06, \\
|j(\tau') - \beta_2| &\leq 6.4 \cdot 10^7|\delta| + 3.11 \cdot 10^5 \leq 3.2 \cdot 10^5.
\end{aligned}$$

For our bound on $\Phi_2(\tilde{j}, z)$ evaluated at $j(\tau')$, we have

$$\begin{aligned}
\left|\Phi_2(\tilde{j}, j(2\tau_0))\right| &= |j(\tau') - \beta_0||j(\tau') - \beta_1||j(\tau') - \beta_2| \\
&\leq 2.8 \cdot 10^{13}|\delta|^2.
\end{aligned}$$

For the first derivative, we have

$$\begin{aligned}
\left|\Phi_2'(\tilde{j}, j(2\tau_0))\right| &= (j(\tau') - \beta_0)(j(\tau') - \beta_1) + (j(\tau') - \beta_0)(j(\tau') - \beta_2) + (j(\tau') - \beta_1)(j(\tau') - \beta_2) \\
&\geq |j(\tau') - \beta_1||j(\tau') - \beta_2| - |j(\tau') - \beta_0||j(\tau') - \beta_1| - |j(\tau') - \beta_0||j(\tau') - \beta_2| \\
&\geq 7.17 \cdot 10^6 \cdot 2.39 \cdot 10^5|\delta| - 12.1 \cdot 7.26 \cdot 10^6|\delta|^2 - 12 \cdot 3.11 \cdot 10^5|\delta| \\
&\geq 1.7 \cdot 10^{12}|\delta|,
\end{aligned}$$

and for the second derivative,

$$\begin{aligned}
\left|\Phi_2''(\tilde{j}, j(\tau'))\right| &\leq 2|j(\tau') - \beta_0| + 2|j(\tau') - \beta_1| + 2|j(\tau') - \beta_2| \\
&\leq 6.5 \cdot 10^5.
\end{aligned}$$

These now give

$$\frac{|\Phi_2(\tilde{j}, j(2\tau_0))||\Phi_2''(\tilde{j}, j(\tau'))|}{|\Phi_2'(\tilde{j}, j(2\tau_0))|^2} \leq \frac{2.8 \cdot 10^{13}|\delta|^2 \cdot 6.5 \cdot 10^5}{(1.7 \cdot 10^{12}|\delta|)^2} \leq 2^{-17} < \frac{1}{2},$$

and for the condition on r, we have

$$\begin{aligned} r &= 35|\delta|, \\ 2\eta &= 2\frac{|\Phi_2(\tilde{j}, j(2\tau_0))|}{|\Phi_2'(\tilde{j}, j(2\tau_0))|} \leq 2\frac{2.8 \cdot 10^{13}|\delta|^2}{1.7 \cdot 10^{12}|\delta|} \leq 34|\delta|, \end{aligned}$$

which ensures convergence. For the rate of convergence, we need a lower bound on the second derivative and an upper bound on the first derivative, which we have as follows,

$$\begin{aligned} \left|\Phi_2''(\tilde{j}, j(\tau'))\right| &\geq 2|j(\tau') - \beta_2| - 2|j(\tau') - \beta_0| - 2|j(\tau') - \beta_1| \\ &\geq 4.7 \cdot 10^5, \end{aligned}$$

and

$$\begin{aligned} \left|\Phi_2'(\tilde{j}, j(2\tau_0))\right| &\leq |j(2\tau_0) - \beta_1||j(2\tau_0) - \beta_2| + |j(2\tau_0) - \beta_0||j(2\tau_0) - \beta_1| \\ &\quad + |j(2\tau_0) - \beta_0||j(2\tau_0) - \beta_2| \\ &\leq 7.25 \cdot 10^6 \cdot 3.11 \cdot 10^5|\delta| + 12.1 \cdot 7.25 \cdot 10^6|\delta|^2 + 12.1 \cdot 3.11 \cdot 10^5|\delta| \\ &\leq 2.3 \cdot 10^{12}|\delta|, \end{aligned}$$

which gives a bound on the convergence, for $k \geq 1$, of

$$\frac{1}{2^k} 2^{-17 \cdot 2^k} \frac{|\Phi_2'(\tilde{j}, j(2\tau_0))|}{|\Phi_2''(\tilde{j}, j(\tau'))|} \leq 2^{-17 \cdot 2^k}.$$

So in order to obtain an absolute precision of 2^{-P}, $\lceil 2 \log P \rceil$ steps will suffice. By Lemma 9, this approximation to β_0 will then be an approximation to either $j(2\tau)$ or $j\left(-\frac{2}{\tau}\right)$ of relative precision $2^{-P/3+11}$. □

3.2.2 j near 0

We now bound the discrepancy between the roots of $\Phi_2(j, z)$ and $\Phi_2(\tilde{j}, z)$ when $\left|\tau - \frac{1+i\sqrt{3}}{2}\right| \leq 2^{-30}$ and $|\tilde{j}| \geq 2^{-P/2}$.

Lemma 10. *Let $j = j(\tau)$, where $\left|\tau - \frac{1+i\sqrt{3}}{2}\right| \leq 2^{-30}$, and suppose that $\tilde{j}$ is an approximation to j of absolute precision 2^{-P}, with $P \geq 300$, and $\left|\tilde{j}\right| \geq 2^{-P/2}$. Then relative precision of any root of $\Phi_2(\tilde{j}, z)$ to its closest root of $\Phi_2(j, z)$ is at most $2^{-P/3+2}$, and the roots of $\Phi_2(\tilde{j}, z)$ are separated by at least $2^{-P/6+8}$.*

Proof. Firstly, as in the proof of the Lemma 9, with $f(z) = \Phi_2(j, z)$ and $g(z) = \Phi_2(\tilde{j}, z)$, we will bound the size of the coefficients of $f - g$. Let $\tilde{j} = j + \delta$. For the coefficient of z^2, we have

$$|2j\delta + \delta^2 + 1448\delta| \leq 1500|\delta|,$$

for the coefficient of z,

$$|2976\delta j + 1488\delta^2 + 40773375\delta| \leq 4.1 \cdot 10^7|\delta|,$$

and for the constant term

$$\left|3\delta j^2 + 3\delta^2 j + \delta^3 - 16200\delta j - 16200\delta^2 + 8748000000\delta\right| \leq 8.8 \cdot 10^9 |\delta|.$$

Now evaluating $g(z)$ at $j(\tau')$, for $\tau' \in \{2\tau, \frac{\tau}{2}, \frac{\tau+1}{2}\}$, as $|j(\tau')| \leq 60000$, we have

$$\begin{aligned} |g(j(2\tau))| &\leq 60000^2 \cdot 1500|\delta| + 60000 \cdot 4.1 \cdot 10^7 |\delta| + 60000 \cdot 8.8 \cdot 10^9 |\delta| \\ &\leq 6 \cdot 10^{14} |\delta|. \end{aligned}$$

Letting $\beta_0, \beta_1, \beta_2$ be the roots of $g(z)$, we have, as $|\tilde{j}| \geq 2^{-P/2}$, which impels $|j| \geq 2^{-P/2-1}$,

$$\begin{aligned} |\beta_0 - j(\tau_i)||\beta_1 - j(\tau_i)||\beta_2 - j(\tau_i)| &\leq 6 \cdot 10^{14} |\delta| \\ &\leq 2^{-P+50} \end{aligned}$$

Now for any β_i, letting $j(\tau_j)$ be the nearest root of $f(z)$, we have

$$|\beta_i - j(\tau_j)| \leq 2^{-P/3+17}.$$

By Lemma 4, as $|j| \geq 2^{-P/2}$,

$$\left|\tau - \frac{1 + i\sqrt{3}}{2}\right| \geq 2^{-P/6-9},$$

which yields, by Lemma 8, for $i \neq j$,

$$|j(\tau_i) - j(\tau_j)| \geq 2^{-P/6+9}.$$

Similarly to the previous lemma, we now observe that, with β_i the closest root to $j(\tau_i)$,

$$\begin{aligned} |\beta_i - \beta_j| &\geq |j(\tau_i) - j(\tau_j)| - 2 \cdot 2^{-P/3+17} \\ &\geq 2^{-P/6+9} - 2^{-P/3+17} \\ &\geq 2^{-P/6+8} \end{aligned}$$

and so β_i, β_j are distinct. So each root has a unique closest associated root, with relative precision

$$\frac{2^{-P/3+17}}{|j(\tau_j)|} \leq 2^{-P/3+2}.$$

□

Proposition 4. *Let* $j = j(\tau)$*, where* $\left|\tau - \frac{1+i\sqrt{3}}{2}\right| \leq 2^{-31}$*, and* $\tilde{j}$ *be an approximation to* j *of relative precision* 2^{-P} *such that* $|\tilde{j}| \geq 2^{-P/2}$*. Let*

$$\tau_0 = \frac{1 + i\sqrt{3}}{2} + \sqrt[3]{\frac{6\tilde{j}}{j^{(3)}(i)}},$$

where the branch of the cube root is chosen arbitrarily. Then Newton's method applied to $\Phi_2(\tilde{j}, z)$*, with starting point* $j(2\tau_0)$ *will converge to either the root of* $\Phi_2(\tilde{j}, z)$ *closest to* $j(2\tau)$*, the root of* $\Phi_2(\tilde{j}, z)$ *closest to* $j\left(\frac{\tau}{2}\right)$*, or the root of* $\Phi_2(\tilde{j}, z)$ *closest to* $j\left(\frac{\tau+1}{2}\right)$*, and after* $\lceil 2 \log P \rceil$ *steps will produce an approximation to either* $j(2\tau)$*,* $j\left(\frac{\tau}{2}\right)$*, or* $j\left(\frac{\tau+1}{2}\right)$ *of relative precision* $2^{-P/3+3}$*.*

Proof. Let $\tau = \frac{1+i\sqrt{3}}{2} + \delta$. By Lemma 3,

$$\left| j(\tau) - \frac{j^{(3)}\left(\frac{1+i\sqrt{3}}{2}\right)}{6}\delta^3 \right| \leq 0.07|\delta|^3,$$

so that

$$\left| \frac{6\tilde{j}}{j^{(3)}\left(\frac{1+i\sqrt{3}}{2}\right)} - \delta^3 \right| \leq 1.6 \cdot 10^{-6}|\delta|^3.$$

We let $\epsilon = \sqrt[3]{\frac{6\tilde{j}}{j^{(3)}\left(\frac{1+i\sqrt{3}}{2}\right)}}$. Considering the geometry of the cube roots of δ^3, the product of the two furthest from ϵ is at least $|\delta|^2$, so letting the closest be δ_0, we have

$$|\epsilon - \delta_0| \leq 1.6 \cdot 10^{-6}|\delta|.$$

We now show that any branch of the cube root taken for ϵ will result $2\tau_0 - 1$ being very close to one of the $\mathrm{SL}_2(\mathbb{Z})$–equivalent elements of $\mathcal{F}$ to one of 2τ, $\frac{\tau}{2}$, $\frac{\tau+1}{2}$. We have

$$\begin{aligned}\left| \frac{1+i\sqrt{3}}{2} + \epsilon - \left(-\frac{1}{\tau} + 1\right) \right| &= \frac{\left|\tau + i\sqrt{3}\tau + 2 + 2\epsilon\tau - 2\tau\right|}{2|\tau|} \\ &= \frac{\left|-\frac{1}{2} - i\frac{\sqrt{3}}{2} - \delta + i\frac{\sqrt{3}}{2} - \frac{3}{2} + i\sqrt{3}\delta + 2 + (1+i\sqrt{3})\epsilon + 2\epsilon\delta\right|}{2|\tau|} \\ &= \frac{\left|(1+i\sqrt{3})\epsilon - (1-i\sqrt{3})\delta + 2\epsilon\delta\right|}{|2\tau|} \\ &\leq |\epsilon - e^{4i\pi/3}\delta| + \frac{|\delta\epsilon|}{2},\end{aligned}$$

and

$$\begin{aligned}\left| \frac{1+i\sqrt{3}}{2} + \epsilon - \left(-\frac{1}{\tau - 1}\right) \right| &= \frac{\left|\tau - 1 + i\sqrt{3}(\tau - 1) + 2 + 2\epsilon(\tau - 1)\right|}{2|\tau|} \\ &= \frac{\left|-\frac{1}{2} + i\frac{\sqrt{3}}{2} + (1+i\sqrt{3})\delta - i\frac{\sqrt{3}}{2} - \frac{3}{2} + 2 + (-1+i\sqrt{3})\epsilon + 2\epsilon\delta\right|}{2|\tau|} \\ &= \frac{\left|(1-i\sqrt{3})\epsilon - (1+i\sqrt{3})\delta - 2\epsilon\delta\right|}{|2\tau|} \\ &\leq |\epsilon - e^{2i\pi/3}\delta| + \frac{|\delta\epsilon|}{2}.\end{aligned}$$

In particular, the difference between $2\tau_0 - 1$ and the closest of the $\mathrm{SL}_2(\mathbb{Z})$–equivalent elements of $\mathcal{F}$ to 2τ, $\frac{\tau}{2}$, $\frac{\tau+1}{2}$, which we let be τ_1, and the others τ_2, τ_3, is bounded by

$$2\,|\epsilon - \delta_0| + |\delta\epsilon| \leq 3.3 \cdot 10^{-6}|\delta|.$$

Now as the absolute value of the derivative of $j(z)$ is bounded in absolute value by $3.4 \cdot 10^5$ between τ_0 and τ_1, we have

$$|j(2\tau_0) - j(\tau_1)| \leq 1.2|\delta|.$$

Now by the bound of Lemma 10 on the discrepancy of the roots of $\Phi_2(\tilde{j},z)$ and $\Phi_2(j,z)$, and the bounds on the distances between the distinct roots of Lemma 8, letting β_0 be the closest root of $\Phi_2(\tilde{j},z)$ to $j(2\tau_0)$, and β_1,β_2 the other two roots, we collect the relevant bounds for Kantorovich's criterion,

$$\begin{aligned} |j(2\tau_0)-\beta_0| &\leq 1.2|\delta| \\ |j(2\tau_0)-\beta_1|\,,|j(2\tau_0)-\beta_2| &\leq 1.35\cdot 10^6|\delta| \\ |j(2\tau_0)-\beta_1|\,,|j(2\tau_0)-\beta_2| &\geq 1.14\cdot 10^6|\delta|, \end{aligned}$$

and with $r=4|\delta|$, for any τ' such that $|\tau'-2\tau_0|\leq 4|\delta|$, we have, as the derivative of j is bounded in absolute value by $3.4\cdot 10^5$ here,

$$\begin{aligned} |j(\tau')-\beta_0| &\leq 1.37\cdot 10^6|\delta| \\ |j(\tau')-\beta_1|\,,|j(\tau')-\beta_2| &\leq 2.71\cdot 10^6|\delta| \end{aligned}$$

Now we bound the various terms of Kantorovich's criterion. Firstly,

$$\begin{aligned} |\Phi_2(\tilde{j},j(2\tau_0))| &\leq 1.2|\delta|\cdot(1.35\cdot 10^6|\delta|)^2 \\ &\leq 2.2\cdot 10^{12}|\delta|^3, \end{aligned}$$

for the first derivative, we have

$$\begin{aligned} |\Phi_2'(\tilde{j},j(2\tau_0))| &\geq (1.14\cdot 10^6|\delta|)^2 - 2\cdot 1.2|\delta|\cdot 1.35\cdot 10^6|\delta| \\ &\geq 1.2\cdot 10^{12}|\delta|^2, \end{aligned}$$

and for the second derivative,

$$\begin{aligned} |\Phi_2''(\tilde{j},j(\tau'))| &\leq 2\cdot 1.37\cdot 10^6|\delta| + 4\cdot 2.71\cdot 10^6|\delta| \\ &\leq 1.36\cdot 10^7|\delta|, \end{aligned}$$

so that we have

$$\frac{|\Phi_2(\tilde{j},j(2\tau_0))||\Phi_2''(\tilde{j},j(2\tau_0))|}{|\Phi_2'(\tilde{j},j(2\tau_0))|^2} \leq \frac{2.2\cdot 10^{12}|\delta|^3\cdot 1.36\cdot 10^7|\delta|}{(1.2\cdot 10^{12}|\delta|^2)^2} \leq 2^{-15} < \frac{1}{2},$$

and for the condition on r we have

$$\begin{aligned} r &= 4|\delta| \\ 2\eta &= 2\frac{|\Phi_2(\tilde{j},j(2\tau_0))|}{|\Phi_2'(\tilde{j},j(2\tau_0))|} \leq 3.8|\delta|, \end{aligned}$$

ensuring convergence. For the rate of convergence, we have the upper bound on the first derivative as follows,

$$\left|\Phi_2'(\tilde{j},j(2\tau_0))\right| \leq 1.4\cdot 10^{12}|\delta|^2.$$

For the lower bound on the second derivative, we have that

$$|\Phi_2''(\tilde{j},j(\tau'))| = |6j(\tau') - 2(\beta_0+\beta_1+\beta_2)|.$$

Now the second term in the absolute value is the coefficient of z^2 of $\Phi_2(\tilde{j}, z)$, which is equal to

$$-\tilde{j}^2 + 1488\tilde{j} - 162000.$$

By Lemma 3, $|j| \leq \left(\frac{\left|j\left(\frac{1+i\sqrt{3}}{2}\right)\right|}{6} + 0.07\right)|\delta|^3$, and as $|\tilde{j} - j| \leq 2^{-P} \leq |j|^2$, $|\tilde{j}| \leq 1.01|j|$, so

$$|-\tilde{j}^2 + 1488\tilde{j}| \leq 4.13 \cdot 10^8 |\delta|^3 \leq 0.1|\delta|.$$

In particular, $-(\beta_1 + \beta_2 + \beta_3) = 162000 + \theta_1$, where $|\theta_1| \leq 0.1|\delta|$. As $|\delta| \leq 2^{-31}$, $|\tau' - 2\tau_0| \leq 4|\delta|$, and $|2\tau_0 - 1 - i\sqrt{3}| \leq (2 + 3.3 \cdot 10^{-6})|\delta|$,

$$|\tau' - 1 - i\sqrt{3}| \leq 6.1|\delta| \leq 2^{-28},$$

and hence by Lemma 5,

$$|j(\tau') - j(i\sqrt{3})| \geq |j'(i\sqrt{3})\delta| - 1.3|\delta| \geq 334000|\delta|.$$

In particular, as $j(i\sqrt{3}) = 54000$, $j(2\tau_0) = 54000 + \theta_2$, where $|\theta_2| \geq 334000|\delta|$. Now combining these bounds, we have

$$\begin{aligned} |6j(\tau') - 2(\beta_0 + \beta_1 + \beta_2)| &= |324000 + 6\theta_2 - 324000 + 2\theta_1| \\ &\geq 6|\theta_2| - 2|\theta_1| \\ &\geq 6 \cdot 334000|\delta| - 0.2|\delta| \\ &\geq 2 \cdot 10^6|\delta|. \end{aligned}$$

Returning to the convergence, we have

$$\frac{|\Phi_2'(\tilde{j}, j(2\tau_0))|}{|\Phi_2''(\tilde{j}, j(\tau'))|} \leq \frac{1.4 \cdot 10^{12}|\delta|^2}{2 \cdot 10^6|\delta|} \leq 2^{-10},$$

so a bound for the rate of convergence is

$$2^{-15 \cdot 2^k}.$$

In particular, to obtain an absolute precision of 2^{-P}, $[2\log P]$ steps will suffice. The approximant obtained will then, by Lemma 10, be an approximation of one of $j(2\tau), j\left(\frac{\tau}{2}\right)$, or $j\left(\frac{\tau+1}{2}\right)$, of relative precision at least $2^{-P/3+3}$. □

3.2.3 Running time

Now we have shown the time required to obtain an approximation to one of $j(2\tau), j\left(\frac{\tau}{2}\right)$, or $j\left(\frac{\tau+1}{2}\right)$ is $O(M(P)\log P)$, and the obtained value may then be used as an input to Newton's method on the compact set described in the subsequent section. In order to determine which of $j(2\tau), j\left(\frac{\tau}{2}\right), j\left(\frac{\tau+1}{2}\right)$ was computed above, after obtaining an inverse τ_0, we compute $j(2\tau_0)$ and $j\left(\frac{\tau_0}{2}\right)$, and return the argument corresponding to whichever was closest to our initial approximation $\tilde{j}$ to j, after applying elements of $\mathrm{SL}_2(\mathbb{Z})$ to move it to the fundamental domain (which takes $O(P)$ time). In particular we will obtain an approximation of precision at least $2^{-P/3+12}$.

3.3 Newton iteration on the compact set

For τ such that $\tau \in \mathcal{F}$, $\mathrm{Im}(\tau) \leq 3.1$, $|\tau - i| \geq 2^{-32}$, $\left|\tau - \frac{1+i\sqrt{3}}{2}\right| \geq 2^{-32}$, we make use of an algorithm due to Dupont, [7] for the quasilinear evaluation of $j(\tau)$ to relative precision, in order to invert $j(z)$ by the secant method. As we are considering a compact set, there is some fixed precision our starting points for the secant method may be in order to obtain convergence for any τ. We compute the low precision inverses of j by the formula

$$\tau_0 = i\frac{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1728}{\tilde{j}}}\right)}{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1728}{\tilde{j}}}\right)},$$

$$\tau_1 = \tau_0 - \frac{j(\tau_0) - j}{j'(\tau_0)},$$

and we note that as j is bounded away from 1728, the arguments of the Gaussian hypergeometric functions are bounded away from their singularities at 0 and 1, and j' is bounded away from 0, so computation of the two points to a fixed precision takes a fixed amount of time. Upon a failure of convergence (or slow convergence), we may simply increase the precision of our starting points and repeat the process – as there is a uniform bound on the required precision, this requires only constant time. As the secant method converges quadratically, and the evaluation of $j(z)$ via the algorithm of [7] takes $O(M(P)\log P)$ time, the computational complexity of obtaining an approximation of precision 2^{-P+1} requires time $O(M(P)(\log P)^2)$.

A similar analysis to that of Lemma 3 gives a bound of

$$\left|j^k(z_0)\right| \leq 3 \cdot 10^8 \cdot 14^k k!$$

for z_0 in our compact set, so that if $z_1 = z_0 + |\delta|$, with $|\delta| \leq 2^{-150}$,

$$\begin{aligned} |j(z_0) + j'(z_0)\delta - j(z_1)| &\leq 3 \cdot 10^8 \sum_{n=2}^{\infty} |\delta|^n 14^n \\ &\leq 3 \cdot 10^8 \cdot 14^2 \cdot 2^{-150} \cdot |\delta| \sum_{n=0}^{\infty} 2^{-150n} 14^n \\ &\leq 5 \cdot 10^{-35} |\delta|. \end{aligned}$$

It may be verified that $|j'(z)| \geq 10^{-19}$ in our compact set, so

$$|j(z_0) - j(z_1)| \geq |j'(z_0)||\delta| - 5 \cdot 10^{-35}|\delta| \geq 10^{-20}|\delta|,$$

and in particular once we have computed by the secant method τ^* such that $|\tilde{j} - j(\tau^*)| \leq 2^{-P}$, $|j - j(\tau^*)| \leq 3 \cdot 10^8 \cdot 2^{-P}$, and so, with $j(\tau) = j$, and $\tau^* = \tau + \delta$, we have

$$|\delta| \leq 3 \cdot 10^8 \cdot 10^{20} \cdot 2^{-P} \leq 2^{-P+100}.$$

3.4 j very close to 0 or 1728

Now if $|\tilde{j}| \leq 2^{-P/2}$, or $|\tilde{j} - 1728| \leq 2^{-P/3}$, we simply return $\frac{1+i\sqrt{3}}{2}$ or i respectively, and this will, by Lemma 4, be an approximation to the inverse of j of absolute, and as $|\tau| \geq 1$, relative precision $2^{-P/6}$.

4 Testing CM

Firstly, given the input of the j–invariant of an elliptic curve E, its degree d, and a bound on its height $H \geq e^e$, we may bound the maximum discriminant from which j may arise, if E were to have complex multiplication. We first consider $|D| \geq 16$. The principal form of a discriminant has an associated τ of imaginary part $\frac{i\sqrt{|D|}}{2}$, so as $|D| \geq 16$, $M(j(\tau)) \geq e^{\pi\sqrt{|D|}} - 2079 \geq e^{3.13\sqrt{|D|}}$. In particular, as $M(j) \leq H^d$,

$$|D| \leq \frac{d^2(\log H)^2}{9.7}.$$

Now the τ associated to a binary quadratic form of discriminant of absolute value $\leq N$ has real part bounded in height by $2N$, and the square of the imaginary part bounded in height by $4N^2$, so we will determine if the preimage of j is a quadratic irrational satisfying these conditions.

We first bound the degree of $j(z)$ at a quadratic irrational of discriminant D. For a fundamental discriminant D, we have the bound (Proposition 2.2, [11]),

$$h(D) \leq \frac{1}{\pi}\sqrt{|D|}(2 + \log|D|),$$

and in the case of non–fundamental discriminants, by Theorem 7.4 of [5], for an odd prime p,

$$h(p^2D) \leq (p+1)h(D) \leq \frac{p+1}{p\pi}\sqrt{|p^2D|}(2 + \log|D|) \leq \frac{4}{3\pi}\sqrt{|p^2D|}(2 + \log|p^2D|)$$

and for 2, if $D \not\equiv 0(8)$,

$$h(4D) \leq 3h(D) \leq \frac{3}{2\pi}\sqrt{|4D|}(2 + \log|D|) \leq \frac{3}{2\pi}\sqrt{|4D|}(2 + \log|4D|)$$

and if $D \equiv 0(4)$,

$$h(4D) \leq 2h(D) \leq \frac{1}{\pi}\sqrt{|4D|}(2 + \log|D|) \leq \frac{1}{\pi}\sqrt{|4D|}(2 + \log|4D|).$$

In particular, we have the bound

$$h(D) \leq \frac{3}{2\pi}\sqrt{|D|}(2 + \log|D|).$$

For the Mahler measure of $j(\tau)$, as $|j(\tau)| \leq e^{2\pi \mathrm{Im}(\tau)} + 2079 \leq 9.1e^{2\pi \mathrm{Im}(\tau)}$, we have the upper bound in terms of the reduced binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant D,

$$M(j(\tau)) \leq 9.1^{h(D)} \exp\left(2\pi\sqrt{|D|} \sum_{\substack{(a,b,c)\\ \text{reduced}}} \frac{1}{a}\right).$$

The number of times each a may occur is bounded by the number of solutions of $b^2 \equiv d(2a)$, which is bounded by twice the number of distinct divisors $r(a)$ of a. By [2], we have the bound

$$A(x) = 2\sum_{1 \leq a \leq x} r(A) \leq 2x\log x + 0.4x + 2x^{1/2},$$

so that by Abel's summation formula,

$$\sum_{1\leq a\leq h(D)} \frac{r(a)}{a} = \frac{A(h(D))}{h(D)} + \int_1^{h(D)} \frac{A(y)}{y^2} dy$$
$$\leq \frac{(\log h(D))^2}{2} + 1.2\log(h(D)) + 2.2$$
$$\leq 1.3\log|D|^2,$$

yielding a bound of

$$M(j(\tau)) \leq 9.1^{h(D)} e^{5.2\pi\sqrt{|D|}(\log|D|)^2} \leq e^{5.9\pi\sqrt{|D|}(\log|D|)^2}.$$

In particular, by Liouville's inequality, if $j \neq j(\tau)$, where τ is of discriminant D,

$$|j - j(\tau)| \geq 2^{-\frac{3d}{2\pi}\sqrt{|D|}(2+\log|D|)} H^{-\frac{3d}{2\pi}\sqrt{|D|}(2+\log|D|)} e^{5.9\pi d\sqrt{|D|}(\log|D|)^2},$$

and substituting our bound on $|D|$, we obtain the lower bound

$$\exp\left(-30d^2 \log H(\log d + \log\log H)^2\right).$$

Now as $|j| \leq H^d$, letting $j = j(z_0)$, where $z_0 \in \mathcal{F}$, as $|j(z_0)| \geq e^{2\pi \mathrm{Im}(z_0)} - 2079$, we have $\mathrm{Im}(z_0) \leq \frac{d\log H}{2\pi} + 2$. By differentiating the q–expansion of j, and taking an upper bound, (note that the sum of the absolute values of the terms of the tail is decreasing), we have, as $d\log H \geq 0$,

$$\|j'\|_{B(z_0,1)} \leq 2\pi e^{d\log H + 6\pi} + 800 \leq e^{d\log H + 21},$$

so that if $|z_0 - \tau| \leq 1$,

$$|j - j(\tau)| \leq |z_0 - \tau| e^{d\log H + 21}.$$

Combining this with our separation result, noting that $\log d + \log\log H \geq 1$, if

$$|z_0 - \tau| \leq \exp\left(-31d^2 \log H(\log d + \log\log H)^2 - 21\right),$$

then $j = j(\tau)$, and so j is a singular modulus. Now as our method of inverting j from an input of regulated precision 2^{-P} obtains its inverse with relative precision at least $2^{-P/6}$, to obtain a result of absolute precision 2^{-Q}, as $|\tau| \leq \frac{d\log H}{2\pi} + 2$, we will need an input of relative precision $2^{-6Q - 2\log(d\log H + 2)}$. In particular, with an input of relative precision

$$2^{-300d^2 \log H(\log d + \log\log H)^2 - 200},$$

our computed approximation $\tilde{z_0}$ to z_0 will have sufficient precision to determine whether

$$|z_0 - \tau| \leq |z_0 - \tilde{z_0}| + |\tau - \tilde{z_0}|$$

is sufficiently small. In order to obtain our candidate τ, we develop the continued fractions of the real part and the square of the imaginary part of our computed inverse $\tilde{\tau}$ – as the discriminants under consideration are bounded by $|D|$, the distance between the real parts of any two preimages of the singular moduli is at least

$$19^{-1}d^{-4}(\log H)^{-4},$$

and of the squares imaginary parts is at least

$$19^{-2}d^{-8}(\log H)^{-8}.$$

So we develop the continued fractions of the real part of $\tilde{z_0}$ and the square of the imaginary part of $\tilde{z_0}$ until their difference from the convergents is bounded by

$$19^{-3}d^{-8}(\log H)^{-8},$$

for once this holds of the convergents, they will form the only possible quadratic irrational inverse of j. If the height of the real convergent c_r is greater than $\frac{d^2(\log H)^2}{9.7}$, or the height of the square of the imaginary convergent c_i is greater than $\frac{d^4(\log H)^4}{90}$, or the square root of c_i is a rational number, then we may conclude that j is not a singular modulus – otherwise, letting $\tau = c_r + i\sqrt{c_i}$, we may test if $|\tilde{z_0} - \tau|$ satisfies our condition for $j = j(\tau)$. If so, then j is a singular modulus, and otherwise j is not a singular modulus for τ of discriminant $|D| \geq 16$. It is clear that the computational complexity of the calculation of the convergents and other operations in this algorithm is dominated by that of inverting j, so we obtain a running time, with $T = d^2 \log H(\log d + \log\log H)^2$ of

$$O(M(T)(\log T)^2).$$

For testing discriminants with $|D| \leq 16$, we may simply take the list of all τ of discriminant ≤ 16, and test whether j is equal to them. The degree of $j(\tau)$ for such τ is bounded by 2, and its Mahler measure is bounded by $3 \cdot 10^6$, so if

$$|j - j(\tau)| \leq 2^{-2d}H^{-2d}\left(3 \cdot 10^6\right)^{-d} \leq \exp(2d(33 + \log H)),$$

then $j = j(\tau)$. So with an approximation $\tilde{j(\tau)}$ to $j(\tau)$ of absolute precision $2^{-4d(33+\log H)+2}$, then we will be able to establish whether $j = j(\tau)$. By the algorithm to compute j of [7], which requires time $O(M(P)\log P)$ for relative precision of P, as $|j(\tau)| \leq 3 \cdot 10^6$, computing j to the required absolute precision possible in time

$$O(M(d\log H)(\log d + \log\log H)),$$

which again is $O\left(M(T)(\log T)^2\right)$.

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