Linear forms in logarithms and the equation of Catalan

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March 5, 2019

Linear forms in logarithms

After Laurent, it may be verified

$$\log|\Lambda| > -780 \log A_1 \log A_2 \log \left(\frac{b_2}{\log A_1} + \frac{b_1}{\log A_2} \right)^2 \tag{1}$$

When $\log A_1 \log A_2 \log \left(\frac{b_2}{\log A_1} + \frac{b_1}{\log A_2} \right)^2 \geq 64$.

Proposition. *Suppose $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent, > 1, and of height ≥ 3. Suppose further b_1, b_2, b_3 have no comon factor, and let*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3 \tag{2}$$

Then

$$|\Lambda| > -2 \cdot 10^5 \log A_1 \log A_2 \log A_3 (\log B)^2 \tag{3}$$

Or there exists a nontrivial linear relation

$$u_1 b_1 + u_2 b_2 + u_3 b_3 = 0 \tag{4}$$

with $|u_i| \leq 36 (\log B)^{1/3} \log A_i$

Tijdeman's proof

Factorizing, we have the following:

$$y^q = (x-1)(x^{p-1} + \cdots + 1) \tag{5}$$

Let $d = \text{hcf}((x-1), (x^{p-1} + \cdots + 1))$. Then $x \equiv 1(d)$, so $0 \equiv (x^{p-1} + \cdots + 1) \equiv p(d)$.

So $d|p$, and thus $d = 1, p$.

So $x - 1 = p^{\delta_1} \rho^q$, $\delta_1 \in \{-1, 0, 1\}$.

Cases:

$$p \nmid x - 1 \Rightarrow \delta_1 = 0 \tag{6}$$

$$p | x-1, p \nmid x^{p-1} + \cdots + 1 \Rightarrow \delta_1 = 0 \tag{7}$$

$$p | x-1, p | x^{p-1} + \cdots + 1 \Rightarrow \delta_1 = \left\{ \begin{array}{lr} -1, & \text{if } p^{q-1} | x-1 \\ 1, & \text{if } p^{q-1} | x^{p-1} + \cdots + 1 \end{array} \right\} \tag{8}$$

Similarly $y+1=q^{\delta_2}\sigma^p$, $\delta_2 \in \{-1,0,1\}$.

Now if $\rho = 1$ or $\sigma = 1$, the inequalities $p < 780\log y < 780\log q < 780\log p$ (if $p > q$) gives an absolute upper bound on p, and thus for q. Similarly for $p < q$.

Now we have, as $\log(x+1) = \log x + \log(1+1/x) < \log x + \frac{2}{x}$, and assuming $p > q$, (so $y > x$ and $q^{\delta_2}\sigma^p > p^{\delta_1}\rho^q$)

$$|p\log(p^{\delta_1}\rho^q+1) - q\log(q^{\delta_2}\sigma^q-1)| < \frac{1}{x^p}, \frac{1}{y^q} \tag{9}$$

$$|p\log(p^{\delta_1}\rho^q) - q\log(q^{\delta_2}\sigma^q)| < \frac{1}{x^p} + \frac{p}{p^{\delta_1}\rho^q} + \frac{q}{q^{\delta_2}\sigma^p} \tag{10}$$

$$|\delta_1 p\log p - \delta_2 q\log q + pq\log\rho/\sigma| < \frac{3p^2}{\rho^q} \tag{11}$$

$$\tag{12}$$

Now suppose $q > 10\log p$. Then $\rho^{q/2} > \rho^{5\log p} = p^{5\log\rho} \geq 3p^2$.

In addition, dividing through our final inequality by pq, we see (as $p \geq 5, \rho \geq 2$),

$$|\log\rho/\sigma| < \frac{1}{\rho^{q/2}} + \frac{2\log p}{q} < \frac{1}{4} \tag{13}$$

So $|\log\max\{\rho,\sigma\}/\log\rho| < 5/4$

Now (KIT, Mignotte) implies

$$\log|\delta_1 p\log p - \delta_2 q\log q + pq\log\rho/\sigma| \geq -2\cdot 10^5(\log p)^4\log\max\{\sigma,\rho\} \tag{14}$$

So we have

$$-\frac{q}{2}\log\rho \geq -2\cdot 10^5(\log p)^4\log\max\{\sigma,\rho\} \tag{15}$$

$$q \leq 5\cdot 10^5(\log p)^4 \tag{16}$$

As $\sigma > 2, q > 10\log p$, we have $p^5 < 2^q \leq \sigma^q$, and as $\rho/\sigma < e^{5/4}$,

$$\max\{p^{\delta_1}\rho^q+1, \sigma^q\} < \sigma^{q/5+5q/4} < \sigma^{\frac{3}{2}q} \tag{17}$$

We also have (similar to before)

$$|p\log(p^{\delta_1}\rho^q+1) - q\log(q^{\delta_2}\sigma^q-1)| < \frac{1}{y^q} \tag{18}$$

$$|p\log(p^{\delta_1}\rho^q+1) - q\log(q^{\delta_2}\sigma^q)| < \frac{1}{y^q} + \frac{q}{q^{\delta_2}\sigma^p} \tag{19}$$

$$\left|p\log\left(\frac{p^{\delta_1}\rho^q+1}{\sigma^q}\right) - \delta_2 q\log q\right| < \frac{1}{y^q} + \frac{q}{q^{\delta_2}\sigma^p} \tag{20}$$

$$< \frac{2q^2}{\sigma^p} \tag{21}$$

Take $p \geq 32$, then $2q^2 < 2q^2 < 2^{p/2} \leq \sigma^{q/2}$, so that

$$\left| p \log \left(\frac{p^{\delta_1} \rho^q + 1}{\sigma^q} \right) - \delta_2 q \log q \right| < \frac{1}{\sigma^{\frac{p}{2}}} \tag{22}$$

Now we apply a linear form in two logarithms, giving the lower bound

$$\log \left| p \log \left(\frac{p^{\delta_1} \rho^q + 1}{\sigma^q} \right) - \delta_2 q \log q \right| > -780 \log \sigma^{\frac{3}{2}q} \log q \log \left(\frac{p}{\log q} + \frac{q}{\sigma^{\frac{3}{2}q}} \right) \tag{23}$$

$$> -\frac{3}{2} \cdot 5 \cdot 10^5 \cdot 780 (\log p)^5 (15.4 + 4 \log \log p)^2 \log \sigma \tag{24}$$

And hence

$$p < \frac{15}{2} \cdot 10^5 \cdot 780 (\log p)^5 (15.4 + 4 \log \log p)^2 \tag{25}$$

$$< 5.85 \cdot 10^8 (\log p)^5 (15.4 + 4 \log \log p)^2 \tag{26}$$

Yielding an absolute bound for p of $1.67 \cdot 10^{18}$, and for q of $6.2 \cdot 10^{11}$.

Baker's estimate of solutions x, y of $y^q + 1 = x^p$ of $\exp(\exp((5p)^{10} q^{10q^3}))$ gives $n < \exp(\exp(\exp(\exp(30.5))))$ if $n, n+1$ are perfect powers, and the same would go through if one assumed $q > p$.